OPTIMAL RESOURCE ALLOCATION IN DOWNLINK CDMA WIRELESS NETWORKS

Irwan Endrayanto Aluicius

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DISSERTATION

to obtain
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on the authority of the rector magnificus,
prof. dr. H. Brinksma,
on account of the decision of the graduation committee,
to be publicly defended
on Thursday 30 May 2013 at 14.45

by

Irwan Endrayanto Aluicius
born on 28 October 1972
in Klaten, Central Java, Indonesia

Dit proefschrift is goedgekeurd door de promotor,
prof. dr. R. J. Boucherie
prof. dr. J.L van den Berg

en de assistent-promotor,
dr. A.F. Gabor

To my parents
To my wife Ina
To my daughters Venda & Dinda

Remember how for forty years now the LORD, your God,
has directed all your journeying in the desert,
so as to test you by affliction and find out
whether or not it was your intention
to keep his commandments.
(The Bible - Deuteronomy 8:2)

Sakèhing prekara daksangga srana kekuwatan
sing diparingaké déning Sang Kristus marang aku.
(I have the strength for everything through HIM who empowers me.)
(The Bible - Philippians 4:13)

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> Science has in fact two aspects. Day science involves reasoning as articulated as gears, results that have the strength of certainty. Aware of its style, proud of its past, sure of its future, the science of days advances in the light. *Night science, on the contrary, wanders in the dark. It hesitates, stumbles, falls.* Questioning everything, it is searching itself endlessly, combining, associating myriads of hypothesis, assumptions still in the form of vague hunches, projects barely taken shape. Nothing guarantees its successes, its ability to survive the tests of logic and experiments, but sometimes thanks to intuition, instinct and the will to discover, as a lightning it illuminates more than a thousand suns....-**François Jacob**, *The Statue Within*.

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Enschede *Irwan Endrayanto Aluicius*
May 2013

Contents

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List of Symbols & Abbreviations

α	the non-orthogonality factor, *page 5.*
ϵ_i^*	the required energy per bit to interference ratio, *page 5.*
$\lambda(\mathbf{T})$	the Perron-Frobenius (PF) eigenvalue of matrix $\mathbf{T}$, *page 18.*
$(E_b/I_0)_i$	the energy per bit to interference ratio for a user i, *page 4.*
$\mathbf{R}_X$	$(r_1, r_2, \cdots, r_I)$, the rates assigned to users in cell X, *page 43.*
$\mathbf{R}_Y$	$(r_1, r_2, \cdots, r_{L-I})$, the rates assigned to users in cell Y, *page 43.*
$\mathcal{B}$	the set of N base transmitter stations (BTS)., *page 75.*
$\mathcal{F}_1(\mathbf{r})$	the set of feasible power assignments with rate assignment $\mathbf{r}$, *page 77.*
$\mathcal{R}_1$	the set of rates within the allowed range for which a feasible power assignment exists, *page 77.*
$\prec_{\mathbf{P}}$	the received interference ordering, *page 80.*
$\widehat{P^X}$	the uplink received signal of BTS X, *page 8.*
$\widehat{P^Y}$	the uplink received signal of BTS Y, *page 8.*
$l_{i,X}$	the path loss of a user i located at distance d_i from BTS X, *page 4.*
N_0	the thermal noise, *page 4.*
N_0^i	the thermal noise received by mobile i, *page 75.*
P_i	the transmission power towards user i, *page 4.*
r_i	the data rate for a user i, *page 4.*

$V(r_i)$	$\frac{\epsilon_i^* r_i}{W+\epsilon_i^* r_i}$, *page 8.*
x	$\sum_{i \in U_X} P_{iX}$, the total transmitted power in BTS X, *page 65.*
y	$\sum_{i \in U_Y} P_{iY}$, the total transmitted power in BTS Y, *page 65.*
CDMA	Code Division Multiple Access, *page 1.*
FPTAS	Fully Polynomial Time Approximation Scheme, *page 42.*
UMTS	Universal Mobile Telecommunications System, *page 1.*
W	the system chip rate, *page 4.*

Chapter 1

Introduction

1.1 Background - problem description

In the past 10-15 years we have witnessed an enormous growth in the demand for mobile communications ranging from speech only and simple mobile data applications in the early years to full mobile Internet with multimedia applications via smart phones nowadays. To handle the mobile traffic growth and meet the increasing service requirements (higher speeds, etc.) new radio access technologies are deployed. In addition, network operators face the challenge to use the capacity of their installed networks as efficiently as possible.

The third generation Universal Mobile Telecommunications System (UMTS) employes Code Division Multiple Access (CDMA) as the technique of sharing the network capacity among users. In a CDMA system, calls share a common spectrum, their transmissions are separated using (pseudo) orthogonal codes. The impact of multiple calls is an increase in the interference level, that limits the capacity of the system. Therefore, variation of load over space and time, and the inherit capacity restrictions due to scarce resources are fundamental issues in the operation of a wireless CDMA system. Load variation may occur at different time scales that require different solutions. At the operational level (time scale of minutes), load fluctuations occur due to randomness in call generation, call location and call lengths. At this time scale, load balancing is carried out via power and rate assignment as well as a reconfiguration of calls over cells. Managing the scarce resources via power and rate assignment requires an underlying algorithm that is fast enough to adapt to variations at this time scale. This thesis develops mathematical models for characterizing and optimizing capacity of CDMA-based wireless network via power and rate allocation.

1.2 Related work

The joint rate and power assignment problem for CDMA systems has received considerable attention over the past 10-15 years. Due to the complexity of the problem, several restrictions have been made, in order to obtain mathematically tractable models. The most common simplifications are considering a cell in isolation, thus neglecting the interference effects, or assuming some extra properties of rates/powers, like unlimited rates or powers. For a simplified model of a single cell in isolation, downlink power assignment schemes for maximizing the throughput (sum of rates) or minimizing the total power in the cell are proposed in [LMS05, DNZ02, YX03]. In [DNZ02], Duan et al. present a procedure for finding the power and rate allocations that minimizes the total transmit power in one cell. For the downlink most studies are based on pole capacity [Sip02] or based on discrete event or Monte-Carlo simulation leading to time consuming evaluation of feasibility and/or capacity [Sta02]. Resource assignment in a multicell environment is more complex than in a single cell, due to the interference caused by users in adjacent cells. It has been studied in the framework of cell-breathing for fixed data rates, see e.g. the pioneering work of [Han95, Yat95] that consider the uplink, that in the early days of CDMA was considered to be the bottle-neck. In this thesis, we aim for developing analytically tractable models for the joint rate and power assignment problem in the downlink of CDMA systems.

First, we review some related work of Chapter 2 where we develop a model for two cell linear model. We consider the joint rate and power assignment problem under the assumption that all users are using the same known rate. This leads to a model for characterizing downlink and uplink power assignment feasibility. For this, we will make use of the Perron Frobenius theory (see [Sen73]). A similar successful work using Perron Frobenius theory on the uplink was presented in [EE99, Han99, BCP00]. Effective interference models such as developed in [EE99] allow for a characterization of feasibility based on the total number of users only. However, the analysis in [EE99] requires a homogeneous distribution of the users over the network cells. In [Han99], feasibility is characterised via the Perron-Frobenius eigenvalue of an interference matrix of the network state. Unfortunately, for the uplink the PF eigenvalue is not available in closed-form so that it provides only a semi analytical evaluation of the uplink capacity. In Chapter 2, the analytical expression of the Perron-Frobenius eigenvalue is available in closed-form. As in [EvdBB05], we derive a condition for the existence of a feasible power allocation for the downlink when the rates allocated to users are known. The discretized downlink two cell model enables a characterization of downlink power feasibility via the Perron-Frobenius eigenvalue of a suitably chosen matrix.

Next, we review some related works of Chapter 3. The model in Chapter 3 is based on Perron-Frobenius theory. Another approach for joint optimal rate and power allocation, based on the Perron-Frobenius theory, is proposed by Berggren [Ber01] and by O'Neill et al. [OJB03]. Berggren [Ber01] describes a distributed algorithm for assigning base transmitter station (BTS) powers such that the common rate of the users is maximized. In [OJB03], Perron-Frobenius theory is used to design an approximation algorithm for a model with multiple rates, which permits the use of techniques from convex optimization. Both papers assume continuous rates for users. The models in [Ber01, OJB03, EvdBB05] have lead us into the model extension with the rates chosen from a discrete set. The goal is to allocate rates from a discrete and finite set $R = \{R_1, ..., R_K\}$ to the users such that the total utility, i.e., the sum of the utilities of all users, is maximized.

Moreover, in Chapter 3 we develop a model with cell decomposition, which leads to a distributed algorithm for downlink rate allocation. In [LMS06], a distributed algorithm, without considering the status of the other cells, was developed via a dynamic pricing algorithm. In Chapter 3, we include the rate allocation of other cells.

Next, in Chapter 4, we extend the model under a continuous rate assumption. The goal is to assign rates to users, such that the utility of the system is maximized. For this purpose, we do a dimension reduction of the power control matrix, as was done for the uplink (see [Han99, MH01, ZBG03]). Due to the complexity of interference-limited systems, analytical solutions for optimal joint power and rate assignments are scarce. In a game theoretic approach, [ST11] optimize power allocation for a single cell. For continuous rates, for a single cell uplink model [KO09] allows a rate dependent energy per bit to interference ratio. For a multiple cell uplink model, in [DYX09] the maximum minimum-rate under quality of service constraints is considered via a power assignment search method. This is a combinatorial optimization method that is similar to that used in [BSW06] for minimizing the total power in a two cell downlink model with fixed data rates.

In Chapter 5, the last part of this thesis, we address the joint downlink rate and power assignment for maximal total system throughput in a multi-cell CDMA network in an analytical setting. It generalizes the results of [BEG07, LMS05, Mus10, ZOB07] to multiple cells to obtain a full analytical characterization of the optimal power and rate assignment in the downlink of a multi-cell CDMA network. [MKT06] shows that in the optimal rate assignment some mobiles operate at maximum rate while others operate at the minimum rate, and only one mobile operates at an intermediate rate, and [ZOB07] shows that the optimal power assignment in the uplink can be obtained by a greedy procedure, where fairness is guaranteed via interference constraints. Optimizing network performance requires performance measures. In this thesis, the satisfaction of a user in segment $i, i \in \{1, ..., L\}$ is measured by means of a positive utility function $u_i(R_i)$. For a presentation of the utility functions commonly used in the literature see [TAG02]. For the uplink, optimal rate and power assignment strategies to maximize total throughput are considered in [HA07, ZMG11, Mus10, OW99, OZW03, VRM11, SS10] and to maximize the minimum rate to achieve fairness in [DYX09, PJ06]. For the downlink, [LMS05] propose a distributed algorithm for rate and power assignment that maximizes total utility. In [Jav06], Javidi analyzes several rate assignments in the context of the trade-off between fairness and overall throughput. The rates are supposed to be continuous and the algorithms proposed for the rate allocation are based on solving the Lagrangean dual. For a subset of utility functions (i.e. either convex or concave or S-shaped or inverse S-shaped), [LK09, ZMG11] propose a near optimal algorithm for downlink resource assignment problems, where the resource may be the power or the rate of the mobile. The non-convex power allocation problem is solved using particle swarm optimization. Weighted fairness is introduced by assigning weights to each user. A dynamic pricing algorithm to obtain a power assignment that maximizes the total utility of the mobiles for two 1 dimensional cells (mobiles situated e.g. on a highway) is proposed in [ZHJ05]. An iterative linear programming approach for joint power allocation and BTS assignment is considered in [LSM09]. An exact algorithm for rate and power assignment that maximizes total throughput in two cells is presented in [BEG07]. Although it may lead to significant imbalance among the mobiles [SSB10], see e.g., [Tsi11] for the trade-off between fairness and throughput, it is argued in [Lit03] that maximal throughput also results in minimal mean sojourn time (time to handle the call).

1.3 Basic models

Effective interference

CDMA is an interference limited system, therefore the capacity of the system is directly related to the interference level. A common measure of the quality of the transmission is the energy per bit to interference ratio, $(E_b/I_0)_i$, that for a user i is defined as (see e.g. [HT07])

$$\left(\frac{E_b}{I_0}\right)_i = \frac{W}{r_i}\frac{\text{useful signal power of user i}}{\text{interference + thermal noise}}, \tag{1.1}$$

where W is the system chip rate, N_0 is the thermal noise and r_i is the data rate for a user i.

First, let us consider the numerator. The received signal power of user i depends on the transmitted power and the user location. In this thesis, for simplification of the mathematical model, we assume deterministic path loss propagation between a transmitter X and a receiver in segment i of the following form

$$P_i^{'} = P_i l_{i,X}, \tag{1.2}$$

where $l_{i,X}$ depends only on the distance d_i between a user i and BTS X, $P_i^{'}$ is the received power of the user i and P_i is the transmission power towards the user i. If $l_{i,X} = d_i^{-\gamma}$, where $\gamma \geq 0$ is independent of the distance, this model performs reasonably well in flat service areas for $d_i \geq 1$ km (see [ARY95, Hat80]).

Next, let us consider the interference term in the denominator. In a multicell environment, since all users are using the same frequency, interference either comes from users in the same cell, called the intracell interference, $I_{\text{intracell}}$, or comes from users in the neighboring cells, called the intercell interference, $I_{\text{intercell}}$. For downlink intracell interference, non-orthogonality factor α represents how much interference can be reduced by the system within the cell. The value of α is between zero and one, i.e., $0 \leq \alpha \leq 1$, where $\alpha = 0$ means the system is completely non-orthogonal and $\alpha = 1$ means the system is completely orthogonal. The higher the value of non-orthogonality factor α, the lower the intracell interference. For uplink intracell interference, it is generally assumed (see e.g. [EE99, HT07]) that the signals are perfectly orthogonal.

Quality of service

In order to ensure a certain quality of service, the energy per bit to interference ratio of a user i has to be above a prespecified value ϵ_i^*, $(E_b/I_0)_i \geqslant \epsilon_i^*$ (see [EE99]). In the presence of perfect power control, we assume that the energy per bit to interference ratio of a terminal in segment i equals the threshold ϵ_i^*, i.e.,

$$\left(\frac{E_b}{I_0}\right)_i = \epsilon_i^*, \text{ for all users } i. \tag{1.3}$$

For the rest of this thesis, we develop models under the perfect power control assumption.

Next, we discuss the downlink transmit power and the uplink received power model separately. Although these problems are similar in nature, that is power and rate assignment is based on maximising a utility function and is subject to energy per bit to interference ratio constraints, the actual power and rate assignment problems differ. In the downlink a few BTSs transmit to many mobiles, whereas in the uplink many mobiles transmit to a few BTSs. The corresponding sources (locations) for interference are different, resulting in similar but different power and rate assignment problems. As a consequence, some insight from the uplink power and rate assignment are of interest for the downlink problem, but a solution for the uplink does not yield a direct solution for the downlink.

Downlink transmit power

Consider a CDMA wireless system with two BTSs, say cell X and cell Y. Assume that the number of users in the systems is L, where I users are assigned to BTS X and $(L - I)$ users are assigned to BTS Y. Let $l_{i,X}$, respectively $l_{i,Y}$, be the path loss of user i from BTS X, respectively from BTS Y. Let us assume that the location of users in cell X is ordered such that $l_{1,X} < l_{2,X} < ... < l_{I,X}$. Thus, the user 1 with path loss $l_{1,X}$ is located the closest to BTS X, and the user I with path loss $l_{I,X}$ is the furthest to BTS X. And from users in cell Y, let us assume that the location of users is ordered such that $l_{I+1,Y} > l_{I+2,Y} > ... > l_{L,Y}$. Thus, the user L with path loss $l_{L,Y}$ is located closest to BTS Y. Hence, the users path loss from BTS X is $l_{1,X} < l_{2,X} < ... < l_{I,X} < l_{I+1,X} < l_{I+2,X} < ... < l_{L,X}$.

Let r_i be the assigned downlink rate to user i that requires a transmit power P_i from the BTS. Under the described path loss model and a constant thermal noise N_0, the energy per bit to interference ratio of user i assigned to BTS X is

$$\left(\frac{E_b}{I_0}\right)_i^{down} = \frac{W}{r_i}\frac{P_i l_{i,X}}{\alpha l_{i,X}(\sum_{j=1}^{I} P_j - P_i) + l_{i,Y}\sum_{j=I+1}^{L} P_j + N_0}, \tag{1.4}$$

for $i \in \{1, ..., I\}$, where $l_{i,X}$ the path loss from the BTS X to user i and P_i is the transmitted power from the BTS to the user in the cell. Similarly, the energy per bit to interference ratio of a user i assigned to BTS Y is

$$\left(\frac{E_b}{I_0}\right)_i^{down} = \frac{W}{r_i}\frac{P_i l_{i,Y}}{\alpha l_{i,Y}(\sum_{j=I+1}^{L} P_j - P_i) + l_{i,X}\sum_{j=1}^{I} P_j + N_0}, \tag{1.5}$$

for $i \in \{I+1, ..., L\}$, where α is the non-orthogonality factor, $l_{i,Y}$ the path loss from the BTS Y to user i and P_i is the transmitted power from the BTS to the user in the cell. Next, we will derive an explicit formulation of the total transmit power of a BTS given that the user i in the cell is assigned a downlink rate $r_{i.}$

From Eq.(1.3) and Eq.(1.4), the downlink transmit power of BTS X to the user i in cell X, for $i \in \{1, ..., I\}$, is

$$P_i = \alpha \sum_{j=1}^{I} V(r_j)P_j + l_i \sum_{j=I+1}^{L} V(r_j)P_j + V(r_i)l_{i,X}^{-1}N_0, \tag{1.6}$$

where

$$V(r_i) = \frac{\epsilon_i^* r_i}{W + \alpha\epsilon_i^* r_i}, \text{ for } i \in \{1, ..., L\}, \tag{1.7}$$

and

$$l_i = \begin{cases} \frac{l_{i,Y}}{l_{i,X}}, \text{ for } i \in \{1, ..., I\}, \\ \frac{l_{i,X}}{l_{i,Y}}, \text{ for } i \in \{I+1, ..., L\}. \end{cases} \tag{1.8}$$

Similarly, from Eq.(1.3) and Eq.(1.5), we can also express the required transmit powers of BTS Y to the user i in cell Y, for $i \in \{I+1, \cdots, L\}$,

$$P_i = l_i \sum_{j=1}^{I} V(r_j)P_j + \alpha \sum_{j=I+1}^{L} V(r_j)P_j + V(r_i)l_{i,Y}^{-1}N_0. \tag{1.9}$$

Uplink received power

The interference model for the uplink differs from that for the downlink, as a for the uplink many terminals transmit to a few BTSs. In the uplink, the interference is measured by the BTS, hence, it is more appropriate in the uplink to measure the received power in the BTS. Let the received power of a user i in BTS X with pathloss $l_{i,X}$ be P_i^X, then

$$P_i^X = P_i l_{i,X}. \tag{1.10}$$

Moreover, the uplink transmit power of a user is limited, say $P_i \leq P_{\max}$.

Let r_i be the uplink rate for user i in cell X. From (1.3), the uplink energy per bit to interference ratio for the user i assigned to BTS X is

$$\left(\frac{E_b}{I_0}\right)_i = \frac{W}{r_i} \frac{P_i^X}{\left(\sum\limits_{j=1}^{I} P_j^X - P_i^X\right) + \sum\limits_{j=I+1}^{L} l_j P_j^Y + N_0}, \tag{1.11}$$

for $i \in \{1, ..., I\}$.

Similarly for BTS Y, the uplink energy per bit to interference ratio for the user i assigned to BTS Y, given that the uplink rate r_i, is

$$\left(\frac{E_b}{I_0}\right)_i = \frac{W}{r_i} \frac{P_i^Y}{\sum\limits_{j=1}^{I} l_j P_j^X + \left(\sum\limits_{j=I+1}^{L} P_j^Y - P_i^Y\right) + N_0}, \tag{1.12}$$

for $i \in \{I+1, ..., L\}$.

For a user i, $i \in \{1, 2, \cdots, L\}$, under the assumption of perfect power control, in the uplink, the user's terminal is required by the BTS to transmit enough power such that $(E_b/I_0)_i = \epsilon_i^*$, for all users i, $i \in \{1, 2, \cdots, L\}$. Moreover, under the assumption of uplink perfect power control, each BTS requires all terminals in the cell to transmit enough power such that the received signal is the same, i.e., $P_i^X = \widehat{P^X}$ and $P_j^Y = \widehat{P^Y}$ (see e.g. [EE99, HT07]). Hence, from (1.11) and (1.12), we have

$$\epsilon_i^* = \frac{W}{r_i} \frac{\widehat{P^X}}{\widehat{P^X}(I-1) + \widehat{P^Y} \sum\limits_{j=I+1}^{L} l_j + N_0}, \tag{1.13}$$

$$\epsilon_i^* = \frac{W}{r_i} \frac{\widehat{P^Y}}{\widehat{P^X} \sum\limits_{j=1}^{I} l_j + \widehat{P^Y}(L-I-1) + N_0}. \tag{1.14}$$

Then, the uplink received signal, $\widehat{P^X}$, at the BTS X should satisfy

$$\widehat{P^X} = V(r_i)\left(I\widehat{P^X} + \left(\sum_{j=I+1}^{L} l_j\right)\widehat{P^Y} + N_0\right), \tag{1.15}$$

and the uplink received signal, $\widehat{P^Y}$, at the BTS Y should satisfy

$$\widehat{P^Y} = V(r_i)\left(\left(\sum_{j=1}^{I} l_j\right)\widehat{P^X} + (L-I)\widehat{P^Y} + N_0\right), \tag{1.16}$$

where

$$V(r_i) = \frac{\epsilon_i^* r_i}{W + \epsilon_i^* r_i}. \tag{1.17}$$

1.4 Overview and contribution

Chapter 2

In Chapter 2, we develop a model for downlink power assignment in a CDMA-based wireless system. We analyze feasibility of the downlink power assignment in a linear model of two CDMA cells, under the assumption that all downlink users in the system receive the same rate. This is done by discretizing the area between two BTSs into small segments. The model considers the number of users and the users' location in each segment. Then, the power requirements are characterized via a matrix representation. We obtain a closed-form analytical expression of the so-called Perron-Frobenius eigenvalue of that matrix. Based on the Perron-Frobenius eigenvalue, we obtain an explicit decomposition of system and user characteristics. Although the obtained relation is non-linear, it basically provides an effective interference characterisation of downlink feasibility for a fast evaluation of outage and blocking probabilities, and enables a quick evaluation of feasibility. We have numerically investigated blocking probabilities and have found for the downlink that it is best to allocate all calls to a single cell. Moreover, this chapter has also provided a model for determining an optimal cell border in CDMA networks. We have combined the downlink and uplink feasibility models to determine cell borders for which the system throughput, expressed in terms of downlink rates, is maximized.

This chapter is based on the papers:

- [EvB03] A.I. Endrayanto, J.L. van den Berg and R.J. Boucherie. Characterizing CDMA downlink feasibility via effective interference, in *Proceedings 1st International Working Conference on Heteregeneous Networks* - HetNetsÕ03, pp. 62/1-62/10, Ilkley, United Kingdom, 21-23 July 2003.

- [EvdBB05] A.I. Endrayanto, J.L van den Berg, R.J Boucherie, An analytical model for CDMA downlink rate optimization taking into account uplink coverage restrictions, *Performance Evaluation* 59, ISSN: 0166-5316 , February 2005.

Chapter 3

In Chapter 3, we extend the model from Chapter 2. This chapter still considers the two cells linear model where the coverage area is divided into small segments. Previously, we have assumed that all users in the cells are using the same rate, regardless users' location. In this chapter, we differentiate rate allocation based on their location. We assume users in the same segment receive the same rate. The rates are chosen from a discrete set. The goal is to assign rates to users in each segment, such that the utility of the system is maximized.

For each segment the transmit power requirements are characterized via a matrix representation that separates user and system characteristics. Based on the Perron-Frobenius eigenvalue of the matrix, we reduce the downlink rate allocation problem to a set of multiple-choice knapsack problems. The solution of these problems provides an approximation of the optimal downlink rate allocation and cell borders for which the system throughput, expressed in terms of utility functions of the users, is maximized. We have reduced the downlink rate allocation problem into a set of multiple-choice knapsack problems. Thus the rate allocation problem is NP-hard. Thus it is very unlikely that polynomial time algorithms exist (unless P=NP). In this chapter, we design an algorithm that is actually a fully polynomial time approximation scheme (FPTAS) for the rate optimization problem. We have derived a combinatorial algorithm for finding a downlink rate allocation in a CDMA network, that, for $\epsilon > 0$, achieves a throughput of value at least $(1 - \epsilon)$ times the optimum.

The approach in this chapter has several advantages. First, the discrete optimization approach has eliminated the rounding errors due to continuity assumptions of the downlink rates. Using our model, the exact rate that should be allocated to each user can be indicated. Second, the rate allocation approximation we have proposed guarantees that the solution obtained is close to the optimum. Moreover, the algorithm works for very general utility functions. Furthermore, the model in this chapter indicates that the optimal downlink rate allocation can be obtained in a distributed way: the allocation in each cell can be optimized independently, interference being incorporated in a single parameter t.

This chapter is based on the papers:

- [EBB04] A.I. Endrayanto, A.F. Bumb, R.J. Boucherie, A multiple-choice knapsack based algorithm for CDMA downlink rate differentiation under uplink coverage restrictions, in *Proceedings 16th ITC Specialist Seminar*, Antwerp, Belgium, 31 August- 2 September 2004.

- [EBBW04] R.J. Boucherie, A.F. Bumb, A.I. Endrayanto, G.J. Woeginger,A combinatorial approximation for CDMA downlink rate allocation, in *Proceedings 7th INFORMS Telecommunications Conference*, Boca Raton, Florida, United States, March 7-10, 2004.

- [BBEW06] R.J. Boucherie, A.F. Bumb, A.I. Endrayanto, G.J. Woeginger, A combinatorial approximation for CDMA downlink rate allocation, in Ch.14 of *Telecommunications Planning: Innovation in Pricing, Network Design and Management*, ISBN: 978-0-387-29222-5 , Springer, 2006.

Chapter 4

In Chapter 4, we propose a fast and exact joint rate and power allocation algorithm in the downlink of a telecommunication network formed by two cells, where the base stations transmit at limited powers. Thus, we incorporate in our model two important aspects of a CDMA network, namely interference and limited powers. We assume that the rates are continuous and may be chosen from a given interval. Thus, it is a different model than that of the previous chapters. The assumption in this chapter seems realistic, since in a CDMA system data rates may be rapidly modified in accordance with channel conditions, resulting in an average rate that lies in an interval.

First, we have developed a model for the joint rate and power allocation problem. Due to the impact of the interference between users in different cells, this problem is much more difficult then that of the previous chapters, where the model was analysed under unlimited powers. Despite its non-convexity, the optimal solutions can be very well characterized. Second, we have analyzed several properties of the optimal solutions. We proved that the optimal rate allocations are monotonic in a function of the path loss. Based on this property, we show that in the optimal rate allocation, only 3 rates are given to users. Finally, we propose a polynomial time algorithm in the number of users that solves optimally the joint rate and power allocation problem. The results can be extended to non-decreasing utility functions. Moreover, the algorithm can be extended to iteratively solve the rate/power allocation problem in a small number of cells.

This chapter is based on the following paper

- [BEG07] R.J. Boucherie, A.F. Gabor, A.I. Endrayanto, Optimal joint rate and power assignment in CDMA networks, presented in *The 3rd International Conference on Algorithmic Aspects in Information and Management* (AAIM'07), Portland, USA, 6-8 June 2007.

- [BGE07] R.J. Boucherie, A.F. Gabor, A.I. Endrayanto, Optimal joint rate and power assignment in CDMA networks, in *Lecture Notes in Computer Science* , ISBN: 978-3-540-72868-9, Springer-Verlag, 2007, pages 201-210.

Chapter 5

In Chapter 5, we extend the continuous rate model to the multi cell case. We present a full analytical characterization of the optimal joint downlink rate and power assignment for maximal total system throughput in a multi cell CDMA network. The cell model is a planar model, where the cell coverage has a hexagonal shape.

Chapter 5 has three main contributions. First, we provide an explicit and exact characterization of the structure of the optimal rate and power assignment. Second, we give a characterization of the optimal rate assignment in each cell. We prove that in a network with N base transmitter stations (BTSs) either all mobiles have maximum rate, or in k BTSs all mobiles have maximum rate and the other BTSs transmit at maximum power, or $N-1$ stations transmit at maximum power. In the latter case, finding the optimal power for the remaining BTS can be reduced to a discrete problem in which only a discrete set of powers must be considered in the optimization procedure. Third, based on these results, we give an exact algorithm for solving the rate and power assignment problem and a fast and accurate heuristic algorithm for power and rate assignment to achieve maximal downlink throughput in a multi cell CDMA system. Under this heuristic, for a cell with the total transmit power less than the maximum, the intermediate rate is neglected, i.e., the heuristic assigns maximum and minimum rates only. Moreover, the heuristic orders the cells according to a certain criterion and assigns maximum power and rates in this order. It is shown that the heuristic is fast and accurate up to high load.

This chapter is based on the paper:

- [EGB12] R.J. Boucherie, A.F. Gabor, A.I. Endrayanto, Exact and Heuristic Algorithm for Throughput Maximization in MultiCell CDMA (submitted).

The relation between chapters is illustrated in Figure 1.1.

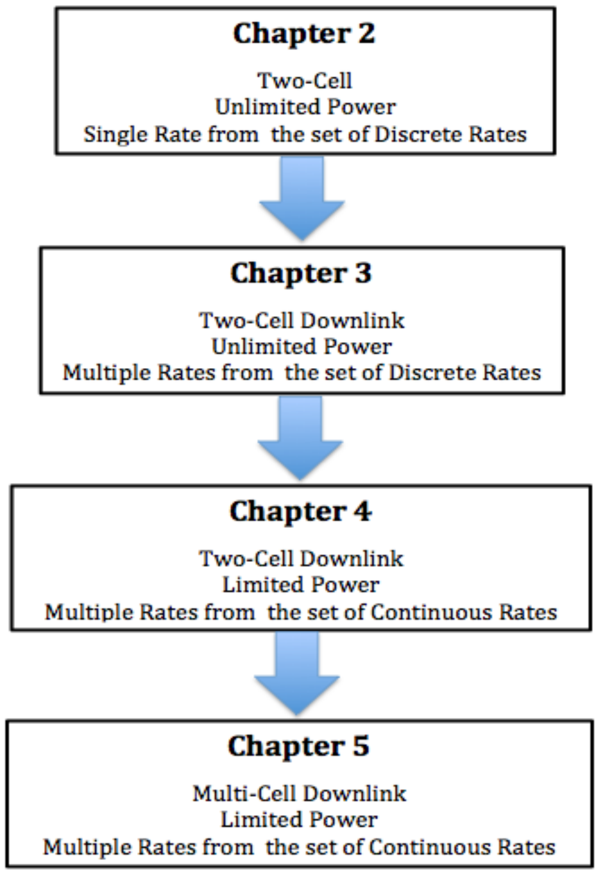


Figure 1.1: *The relation between chapters*

Chapter 2

Characterizing CDMA Feasibility via Effective Interferences

2.1 Introduction

One of the most important features of current wireless communication systems is their support of different user data rates. As a major complicating factor, due to their scarcity, the radio resources have to be used very efficiently. The third generation Universal Mobile Telecommunications System (UMTS) employees Code Division Multiple Access (CDMA) as the technique of sharing the network capacity among multiple users. The impact of multiple simultaneous calls is an increase in the interference level, that limits the capacity of the system. The assignment of transmission powers to calls is an important problem for network operation, since the interference caused by a call is directly related to the power. In the CDMA downlink, the transmission power is related to the downlink rates. Hence, for an efficient system utilization, it is necessary to adopt a rate allocation scheme in the transmission power assignment.

The objective of this chapter is to develop an analytical model that allows a fast evaluation of the downlink feasibility of CDMA under non-homogeneous traffic load. In particular, we aim for an effective interference model. We derive a condition for the existence of a feasible power allocation for the downlink when the rates allocated to users are known. By discretizing the cell into segments, we obtain an analytical model for characterizing the transmit power feasibility for a certain rate allocation and a certain user distribution. Furthermore, we develop a feasibility model that will be used in the later sections for determining the optimal border location and the optimal rate allocation. For this, we will make use of the Perron Frobenius theory (see [Sen73]), by analogy with the characterization of power feasibility for the uplink in [BCP00, EE99, Han99]. Effective interference models such as developed in [EE99] allow for a characterization of feasibility based on that total number only, but they assume a homogeneous distribution of calls over the area covered by a cell.

2.2 Discretized two cell model

We focus on modeling BTSs located along a highway to include both non homogeneity of the user distribution, and mobility of users. Users are located in cars passing through the cells. Due to e.g. traffic jams ("hot spots") the load of the cells will not be distributed evenly along the road. To characterize the distribution of a single type of users in the cells, we propose a discretized-cell model. Each cell is divided into small segments. Then, the non homogeneous load can be characterized by the mean number of calls and fresh call arrival rates in the segments. Taking into account interference between segments in neighboring cells and between segments within the cells, we express the generated downlink interference per segment towards the other segments. This model permits, as we will see below, to characterize analytically the transmit power feasibility for a given rate allocation and user distribution.

We consider a linear network model. Let X and Y be the two base stations (BTSs), situated at distance D from each other on a highway. The highway is divided into L small segments of length δ. For the description below, we fix the radii of the cells. Let cell X contain I segments, labelled as $i = 1, ..., I$, and let cell Y contain $L - I$ segments, labelled as $i = I + 1, ..., L$.

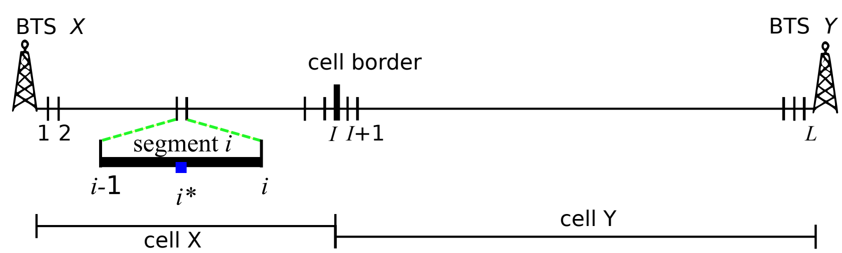


Figure 2.1: *Discretized Cell Model*

We assume that the segments are small, so that we may approximate the location of users in a segment to be in the middle of that segment, i.e. for segment i of cell X, users are located at distance $i^* = \delta\left[(i-1)+i\right]/2$ from X. Furthermore, we also assume that in each segment, the users have the same data rate and power. Denote by n_i the number of users in segment i, for $i \in \{1, ..., L\}$.

2.2.1 Downlink interference model

To characterize the interference first we consider a single user type model, where all users have the same downlink data rate r_d

$$r_i = r_d, \text{ for all } i \in \{1, 2, \cdots, L\}. \tag{2.1}$$

Recall Figure 2.1. Assume that the number of users, n_i, for $i \in \{1, ..., L\}$ in each segment of both BTSs is known. Under the described path loss model, with users in the same segment having the same power and the same rate and a constant thermal noise N_0, the energy per bit to interference ratio in the segments assigned to BTS X becomes

$$\left(\frac{E_b}{I_0}\right)_i^{down} = \frac{W}{r_d} \frac{P_i l_{i,X}}{\alpha l_{i,X}(\sum\limits_{j=1}^{I} P_j n_j - P_i) + l_{i,Y} \sum\limits_{j=I+1}^{L} P_j n_j + N_0}, \tag{2.2}$$

for $i \in \{1, ..., I\}$. And, the energy per bit to interference ratio in the segments assigned to BTS Y ia

$$\left(\frac{E_b}{I_0}\right)_i^{down} = \frac{W}{r_d} \frac{P_i l_{i,Y}}{\alpha l_{i,Y}(\sum\limits_{j=I+1}^{L} P_j n_j - P_i) + l_{i,X} \sum\limits_{j=1}^{I} P_j n_j + N_0}, \tag{2.3}$$

for $i \in \{I+1, ..., L\}$, where α is the non-orthogonality factor.

From Chapter 1, under the assumption of perfect power control we have

$$\left(\frac{E_b}{I_0}\right)_i^{down} = \epsilon_d^*. \tag{2.4}$$

Downlink transmit power

From Eq.(2.2) and Eq.(2.4), we express the explicit formulation of downlink transmit power of BTS X to the user in segment i, for $i \in \{1, ..., I\}$ is

$$P_i = \alpha V(r_d) \sum_{j=1}^{I} P_j n_j + V(r_d) l_i \sum_{j=I+1}^{L} P_j n_j + V(r_d) l_{i,X}^{-1} N_0, \tag{2.5}$$

where

$$V(r_d) = \frac{\epsilon_d^* r_d}{W + \alpha \epsilon_d^* r_d}, \text{ for } i \in \{1, ..., L\}, \tag{2.6}$$

and

$$l_i = \begin{cases} \frac{l_{i,Y}}{l_{i,X}}, & \text{for } i \in \{1, ..., I\}, \\ \frac{l_{i,X}}{l_{i,Y}}, & \text{for } i \in \{I+1, ..., L\}. \end{cases} \tag{2.7}$$

Similarly, from Eq.(2.3) and Eq.(2.4), we can also express the required transmit powers of BTS Y to all users in segment i, for $i \in \{I+1, \cdots, L\}$.

$$P_i = V(r_d) l_i \sum_{j=1}^{I} P_j n_j + \alpha V(r_d) \sum_{j=I+1}^{L} P_j n_j + V(r_d) l_{i,Y}^{-1} N_0. \tag{2.8}$$

2.2.2 Persistent calls model

In this section, we develop a model for persistent calls to analyze interference in CDMA based on the cell model in Figure 2.1. The objective is to characterize analytically the transmit power feasibility for a given rate allocation and users distribution.

Characterization of solution

From Equation (2.5) and (2.8), the solution of downlink transmit powers for both cells can be found by solving the following system of equations

$$\begin{cases} P_i = V(r_d) \left(\alpha \sum\limits_{j=1}^{I} P_j n_j + l_i \sum\limits_{j=I+1}^{L} P_j n_j + l_{i,X}^{-1} N_0 \right), \\ \qquad \text{for } i \in 1, ..., I, \\ P_i = V(r_d) \left(l_i \sum\limits_{j=1}^{I} P_j n_j + \alpha \sum\limits_{j=I+1}^{L} P_j n_j + l_{i,Y}^{-1} N_0 \right), \\ \qquad \text{for } i \in I+1, ..., L, \\ P_i \geq 0, \ \text{for } i \in 1, ..., L. \end{cases} \tag{2.9}$$

Note that system (2.9) has L equations, besides the positivity constraint of the power vector. We can write the system of equations into matrix form as follows

$$(\mathbf{I} - \mathbf{T}) \mathbf{P} = \mathbf{c}. \tag{2.10}$$

where $\mathbf{I}$ is an identity matrix of size $(L \times L)$, $\mathbf{P} = \begin{pmatrix} P_1 & \cdots & P_I & P_{I+1} & \cdots & P_L \end{pmatrix}^T$ is the BTSs transmit power represented in a vector column of size $(L \times 1)$, $\mathbf{c} = V(r_d)\ N_0 \begin{pmatrix} l_{1,X}^{-1} & \cdots & l_{I,X}^{-1} & l_{I+1,Y}^{-1} & \cdots & l_{L,Y}^{-1} \end{pmatrix}^T$ is the vector column of size $(L \times 1)$ related to own interference of thermal noise. Matrix $\mathbf{T}$ characterizes the interference related to the number of users and their locations, which can be written in block matrices

$$\mathbf{T} = V(r_d) \left(\begin{array}{c|c} \alpha \mathbf{1}_{I \times I} & \mathbf{L}_X^Y \\ \hline \mathbf{L}_Y^X & \alpha \mathbf{1}_{(L-I) \times (L-I)} \end{array} \right) \left(\begin{array}{c|c} \mathbf{U}_X & \mathbf{0}_{I \times (L-I)} \\ \hline \mathbf{0}_{(L-I) \times I} & \mathbf{U}_Y \end{array} \right), \tag{2.11}$$

where

- $\mathbf{1}_{I\times I}$, respectively $\mathbf{1}_{(L-I)\times(L-I)}$, is a matrix of size $(I\times I)$, respectively $(L-I)\times(L-I)$, with all elements equal to one.
- $\mathbf{0}_{I\times(L-I)}$, respectively $\mathbf{0}_{(L-I)\times -I}$, is a matrix of size $I\times(L-I)$, respectively $(L-I)\times I$, with all elements equal to zero.
- $\mathbf{L}_X^Y$ is a matrix of size $I\times(L-I)$,represents the fraction of pathloss (see Eq.(2.7)) of users located in I segments of BTS X to users located in $(L-I)$ segments of BTS Y. As each row represent a position in the i^{th} segment, the element for the i^{th} row is equal to l_i, i.e.,

  $$\mathbf{L}_X^Y(i,j)=l_i,\ \text{for } i=1,2,\cdots,I.$$

  $\mathbf{L}_Y^X$ is a matrix of size $(L-I)\times I$, that represents the fraction of pathloss of users located in $(L-I)$ segments of BTS Y to users located in I segments of BTS X. Similarly to above, as each row represent a position in the i^{th} segment, the element for the i^{th} row is equal to l_i, i.e.,

  $$\mathbf{L}_Y^X(i,j)=l_i,\ \text{for } i=I+1,\cdots,L.$$

- $\mathbf{U}_X$, a diagonal matrix of size $(I\times I)$ with element $U_X(i,i)=n_i$, represent the number of users in segments of BTS X and $\mathbf{U}_Y$, a diagonal matrix of size $(L-I)\times(L-I)$, $U_Y(i,i)=n_i$, represent the number of users in segments of BTS Y. Thus,

  $$\begin{array}{l}\mathbf{U}_X=(n_1,n_{2,}\ldots,n_I)^T\mathbf{I}_{I\times I},\\ \mathbf{U}_Y=(n_{I+1},n_{I+2,}\ldots,n_L)^T\mathbf{I}_{(L-I)\times(L-I)},\end{array} \tag{2.12}$$

  where $\mathbf{I}_{I\times I}$, respectively $\mathbf{I}_{(L-I)\times(L-I)}$, is an identity matrix of size $(I\times I)$, respectively $(L-I)\times(L-I)$.

Thus, downlink transmit power feasibility of our cellular system is characterized by the matrix **T,** where the distribution of calls over the segments appears in **T**. The system and user characteristics in this matrix can be separated as in (2.11), which can be rewritten as

$$\mathbf{T}=\mathbf{S}\mathbf{U}, \tag{2.13}$$

where **S** represents the system parameters

$$\mathbf{S}=\left(\begin{array}{c|c}\alpha V(r_d)\mathbf{1}_{I\times I} & V(r_d)\mathbf{L}_X^Y\\ \hline V(r_d)\mathbf{L}_Y^X & \alpha V(r_d)\mathbf{1}_{(L-I)\times(L-I)}\end{array}\right), \tag{2.14}$$

and **U** represents the distribution of the number of calls in each segment

$$\mathbf{U}=\left(\begin{array}{c|c}\mathbf{U}_X & \mathbf{0}_{I\times(L-I)}\\ \hline \mathbf{0}_{(L-I)\times I} & \mathbf{U}_Y\end{array}\right). \tag{2.15}$$

Note that the entries of **S** are fixed for given system parameters. Thus the solution of downlink transmit powers is determined by the distribution of calls over the segments.

Feasible solution

The solution of downlink transmit powers in Eq.(2.9) can be found by solving the system of equations in Eq.(2.10). Since matrix $\mathbf{T}$ is a non-negative matrix, according to the Perron-Frobenius theorem (see [Sen73]), the feasibility of (2.10) is determined by the Perron- Frobenius (PF) eigenvalue $\lambda(\mathbf{T})$ of the matrix $\mathbf{T}$. For the sake of completeness, we present the Perron-Frobenius theorem below.

Theorem 2.2.1 *[Sen73] A necessary and sufficient condition for a solution* $\mathbf{P}$ $(\mathbf{P} \geq \mathbf{0}, \neq \mathbf{0})$ *to the equations*

$$(s\mathbf{I} - \mathbf{T})\mathbf{P} = \mathbf{c} \tag{2.16}$$

to exist for any $\mathbf{c} \geq \mathbf{0}, \neq \mathbf{0}$ *is that* $\lambda(\mathbf{T}) < s$.
In this case there is only one solution of $\mathbf{P}$*, which is strictly positive and given by*

$$\mathbf{P} = (s\mathbf{I} - \mathbf{T})^{-1}\mathbf{c}$$

For our model in this chapter we have $s = 1$. Thus

$$\mathbf{P} \geq \mathbf{0} \text{ exist and } \mathbf{P} = (\mathbf{I} - \mathbf{T})^{-1}\mathbf{c} \quad \Longleftrightarrow \quad \lambda(\mathbf{T}) < 1. \tag{2.17}$$

Next theorem gives the explicit formulation of the Perron- Frobenius (PF) eigenvalue $\lambda(\mathbf{T})$.

Theorem 2.2.2 *The Perron-Frobenius* (PF) *eigenvalue of* $\mathbf{T}$ *is*

$$\begin{aligned} \lambda(\mathbf{T}) &= \frac{V(r_d)}{2}\left(\sum_{i=1}^{I}\alpha n_i + \sum_{j=I+1}^{L}\alpha n_j\right) \\ &\quad + \frac{V(r_d)}{2}\sqrt{\left(\sum_{i=1}^{I}\alpha n_i - \sum_{j=I+1}^{L}\alpha n_j\right)^2 + 4\sum_{i=1}^{I}n_i l_i \sum_{j=I+1}^{L} V n_i l_i}\quad . \end{aligned} \tag{2.18}$$

Proof. The PF eigenvalue of matrix T is determined from the characteristic polynomial of matrix $\mathbf{T}$, i.e., $|\mathbf{T}-\lambda\mathbf{I}| = 0$. As $\mathbf{T}= \mathbf{SU}$, we find

$$|\mathbf{T}-\lambda\mathbf{I}| = \left|\mathbf{S}-\lambda\mathbf{I}\mathbf{U}^{-1}\right| |\mathbf{U}| . \tag{2.19}$$

$\mathbf{U}$ is a diagonal matrix so that $\det(\mathbf{U})$ is the multiplication of the diagonal elements, i.e.,

$$|\mathbf{U}| = \prod_{i=1}^{I} n_i \prod_{j=I+1}^{L} n_j . \tag{2.20}$$

Hence, it remains to calculate $\left|\mathbf{S}-\lambda\mathbf{U}^{-1}\mathbf{I}\right|$. Notice that $\left|\mathbf{S}-\lambda\mathbf{U}^{-1}\mathbf{I}\right|$ has a block matrix structure,

$$\left|\mathbf{S}-\lambda\mathbf{U}^{-1}\mathbf{I}\right| = \begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix}.$$

For block matrices with $\det(\mathbf{A}) \neq 0$, the determinant is (see [Mey00])

$$\det\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \det(\mathbf{A})\det\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right).$$

Straight forward algebra gives

$$\left|\mathbf{S}-\lambda\mathbf{U}^{-1}\mathbf{I}\right| = (-\lambda)^{(L-2)}\left(\prod_{i=1}^{I}\frac{1}{n_i}\right)\left(\prod_{j=I+1}^{L}\frac{1}{n_j}\right)\times K, \tag{2.21}$$

where

$$K = \left(\alpha V(r_d)\sum_{j=I+1}^{L} n_j - \lambda\right)\left(\alpha V(r_d)\sum_{i=1}^{I} n_i - \lambda\right) - (V(r_d))^2\sum_{i=1}^{I} l_i n_i \sum_{j=I+1}^{L} l_j n_j.$$

Hence, from (2.20) and (2.21)

$$\begin{aligned}
|\mathbf{T}-\lambda\mathbf{I}| &= \left|\mathbf{S}-\lambda\mathbf{I}\mathbf{U}^{-1}\right||\mathbf{U}|, \\
&= \det(\mathbf{A})\det\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)\det(\mathbf{U}), \\
&= (-\lambda)^{(I+J-2)}F(\lambda),
\end{aligned}$$

where

$$\begin{aligned}
F(\lambda) = \lambda^2 &- \lambda\alpha V(r_d)\left(\sum_{i=1}^{I} n_i + \sum_{j=I+1}^{L} n_j\right) \\
&+ (V(r_d))^2\left(\sum_{i=1}^{I}\alpha n_i \sum_{j=I+1}^{L}\alpha n_j - \sum_{i=1}^{I} l_i n_i \sum_{j=I+1}^{L} l_j n_j\right).
\end{aligned}$$

Clearly $|\mathbf{T}-\lambda\mathbf{I}| = 0$ has $(L-2)$ zero eigenvalues and only two non-zero eigenvalues. These eigenvalues are determined from the solution of $F(\lambda) = 0$. Thus, the Perron-Frobenius eigenvalue of $\mathbf{T}$ is the largest root of $F(\lambda) = 0$ as in Eq.(2.18). ■

The characterization of downlink feasibillity via matrix $\mathbf{T}$, the feasibility solution via Perron-Frobenius theory and the explicit formulation of $\lambda(\mathbf{T})$ in (2.18), provide a clear motivation for the discretization into segments as we obtain a downlink interference model.

Next, for the sake of completeness, although the uplink interference model has been studied extensively in [BCP00, EE99, Han99], we present it briefly in our setting of the discretized cell model. This uplink model will be used in combination with the downlink model.

2.2.3 Uplink interference model

In a CDMA system the uplink (mobile user to Base Transmitter Station (BTS)) and downlink (BTS to mobile) have different characteristics, and must be analyzed separately. The uplink determines coverage, whereas the downlink determines capacity. As the downlink has more capacity (due to e.g. a higher transmit power of the BTSs), in many studies the uplink has been investigated in detail.

A successful analytical uplink concept is the effective interference model developed by [EE99], which enables a fast evaluation of network state feasibility. However, the analysis in [EE99] requires a homogeneous distribution of the users over the network cells. In [Han99], feasibility is characterized via the Perron-Frobenius eigenvalue of an interference matrix of the network state. Unfortunately, for the uplink the PF eigenvalue is not available in closed-form so that it provides only a semi analytical evaluation of the uplink capacity. For the downlink most studies are based on pole capacity [Sip02] or based on discrete event or Monte-Carlo simulation leading to slow evaluation of feasibility and/or capacity [Sta02].

The interference model for the uplink is different than that for the downlink, as a for the uplink many users transmit to a few BTSs. In the uplink, the interference is measured by the BTS, hence, it is more appropriate in the uplink to measure the received power in the BTS.

Let the received power of a user i in BTS X at path loss $l_{i,X}$ be P_i^X, then

$$P_i^X = P_i l_{i,X}. \tag{2.22}$$

Moreover, the uplink transmit power of a user is limited, say $P_i \leq P_{\max}$. As in downlink, it is assumed that the uplink data rate is the same r_u for all users, i.e.,

$$r_i = r_u, \text{ for all } i \in \{1, 2, \cdots, L\}. \tag{2.23}$$

From (1.1), the uplink energy per bit to interference ratio in the segments assigned to BTS X is

$$\left(\frac{E_b}{I_0}\right)_i = \frac{W}{r_u} \frac{P_i^X}{\left(\sum_{j=1}^{I} n_j P_j^X - P_i^X\right) + \sum_{j=I+1}^{L} l_j n_j P_j^Y + N_0}, \tag{2.24}$$

for $i \in \{1, ..., I\}$.

Similarly for BTS Y, the uplink energy per bit to interference ratio in the segments assigned to BTS Y is

$$\left(\frac{E_b}{I_0}\right)_i = \frac{W}{r_i} \frac{P_i^Y}{\sum_{j=1}^{I} l_j n_j P_j^X + \left(\sum_{j=I+1}^{L} n_j P_j^Y - P_i^Y\right) + N_0}, \tag{2.25}$$

for $i \in \{I+1, ..., L\}$.

For a user in segment i, $i \in \{1, 2, \cdots, L\}$, under the assumption of perfect power control, in the uplink, a user in segment i is required by the BTS to transmit enough power such that $(E_b/I_0)_i = \epsilon_u^*$, for all uplink connection of a user in segment i, $i \in \{1, 2, \cdots, L\}$. Moreover, under the assumption of uplink perfect power control each BTS requires all users in the cell to transmit enough power such that the received signal is the same, i.e., $P_i^X = \widehat{P^X}$ and $P_j^Y = \widehat{P^Y}$ (see e.g. [EE99, HT07]). Hence, from (2.24) and (2.25), we have

$$\epsilon_u^* = \frac{W}{r_u} \frac{\widehat{P^X}}{\widehat{P^X}\left(\sum\limits_{j=1}^{I} n_j - 1\right) + \widehat{P^Y} \sum\limits_{j=I+1}^{L} l_j n_j + N_0}, \tag{2.26}$$

$$\epsilon_u^* = \frac{W}{r_u} \frac{\widehat{P^Y}}{\widehat{P^X} \sum\limits_{j=1}^{I} l_j n_j + \widehat{P^Y}\left(\sum\limits_{j=I+1}^{L} n_j - 1\right) + N_0}. \tag{2.27}$$

Then, the uplink received signal, $\widehat{P^X}$, at the BTS X should satisfy

$$\widehat{P^X} = V(r_u)\left(\widehat{P^X}\left(\sum_{j=1}^{I} n_j\right) + \widehat{P^Y} \sum_{j=I+1}^{L} l_j n_j + N_0\right), \tag{2.28}$$

and the uplink received signal, $\widehat{P^Y}$, at the BTS Y should satisfy

$$\widehat{P^Y} = V(r_u)\left(\widehat{P^X}\left(\sum_{j=1}^{I} l_j n_j\right) + \widehat{P^Y}\left(\sum_{j=I+1}^{L} n_j\right) + N_0\right), \tag{2.29}$$

where

$$V(r_u) = \frac{\epsilon_u^* r_u}{W + \epsilon_u^* r_u}. \tag{2.30}$$

Rewriting (2.28) and (2.29) into matrix form, we have

$$(\mathbf{I} - \widehat{\mathbf{T}})\widehat{\mathbf{P}} = \widehat{\mathbf{c}}, \tag{2.31}$$

where

$$\widehat{\mathbf{T}} = V(r_u)\begin{pmatrix} \left(\sum\limits_{j=1}^{I} n_j\right) & \sum\limits_{j=I+1}^{L} l_j n_{jj} \\ \left(\sum\limits_{j=1}^{I} l_j n_j\right) & \left(\sum\limits_{j=I+1}^{L} n_j\right) \end{pmatrix}, \tag{2.32}$$

$$\widehat{\mathbf{P}} = \begin{pmatrix} X^R \\ Y^R \end{pmatrix}, \quad \widehat{\mathbf{c}} = V(r_u) N_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Uplink feasibility

Uplink feasibility via the PF eigenvalue of $\widehat{\mathbf{P}}$ was investigated in [Han99], where the condition

$$\widehat{\mathbf{P}} \geq 0 \text{ exist and } \widehat{\mathbf{P}} = (\mathbf{I} - \widehat{\mathbf{T}})^{-1}\widehat{\mathbf{c}} \Longleftrightarrow \quad \widehat{\lambda}(\widehat{\mathbf{T}}) < 1. \tag{2.33}$$

was used. An explicit expression for the PF eigenvalue, however, was not provided. Theorem below provides this expression. As the proof is straightforward, it is omitted.

Theorem 2.2.3 *The Perron-Frobenius eigenvalue of* $\widehat{\mathbf{T}}$ *is*

$$\begin{aligned} \widehat{\lambda}\left(\widehat{\mathbf{T}}\right) \quad &= \quad \frac{V(r_u)}{2}\left(\sum_{i=1}^{I} n_i + \sum_{j=I+1}^{L} n_j\right) \\ &+ \frac{V(r_u)}{2}\sqrt{\left(\sum_{i=1}^{I} n_i - \sum_{j=I+1}^{L} n_j\right)^2 + 4\left(\sum_{i=1}^{I} n_i l_i\right)\left(\sum_{i=I+1}^{L} n_i l_i\right)} \quad . \end{aligned} \tag{2.34}$$

2.2.4 Non-persistent calls model

In previous section, we have established downlink and uplink feasibility for persistent calls via the Perron-Frobenius (PF) eigenvalues of the matrix $\mathbf{T}$ and $\widehat{\mathbf{T}}$, that is explicitly provided in Theorem 2.2.2 and Theorem 2.2.3 considers the model with non-persistent calls and discusses the time-dependent distribution of calls over the segments, and corresponding blocking and outage probabilities.

Feasible region

By discretizing the cell, we obtain an explicit expression for the PF eigenvalue of $\mathbf{T}$ that can be used to characterize the feasibility of the downlink connection for a non-homogeneous distribution of calls over the segments. Using the explicit formulation of the PF eigenvalue in Eq.(2.18), the feasibility of a user configuration $\mathbf{U}$ is now readily determined by checking the inequality $\lambda(\mathbf{T}) = \lambda(\mathbf{U}) < 1$. The set of all feasible user configurations is

$$\mathcal{S}_D = \left\{\mathbf{U} \mid \lambda(\mathbf{U}) < 1, \quad \mathbf{U} = \in \mathbb{N}^L\right\}. \tag{2.35}$$

It can readily be shown that $\mathcal{S}_D$ is a coordinate convex set, so that we may invoke the theory of loss networks [Ros95] to characterize the distribution of non-persistent calls.

As in the downlink, we define the set of all feasible user configurations in the uplink. Thus, by developing a discretized cell model, we are able to derive an explicit formulation of the PF eigenvalue not only for the downlink but also for the uplink. If we compare the downlink and uplink feasibility, we see that the expression for $\widehat{\lambda}\left(\widehat{\mathbf{T}}\right)$ is similar to $\lambda\left(\mathbf{T}\right)$ for $\alpha = 1$. Thus, when there is no downlink interference reduction, i.e., the non-orthogonality factor is equal to 1 or it is completely non-orthogonal, the interference in the downlink is similar to the uplink:

$$\mathcal{S}_U = \left\{ \mathbf{U} \mid \widehat{\lambda}\left(\mathbf{U}\right) < 1, \quad \mathbf{U} \in \mathbb{N}^L \right\}. \tag{2.36}$$

This is also a coordinate convex set.

Moving calls

Consider the discretized linear wireless network with non-persistent and moving users. Let fresh calls arrive according to a Poisson arrival process with rate proportional to the density of users along the road, and let users move along the road according to the laws of road traffic movement.

The prediction of the location of subscribers used in this paper requires an estimate of the density of users. For the purpose of this paper, a simplified model as provided in [New93] is sufficient. Let $k(x,t)$ denote the density of users at location x at time t. Then the traffic mass conservation principle states that

$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial k(x,t)v(x,t)}{\partial x} = 0, \tag{2.37}$$

where $v(x,t)$ is the velocity on location x at time t.

In a mobile network the number of users making a call is typically substantially smaller than the number of users not making a call. Therefore, it is natural to assume that *fresh calls* in segment i are generated according to a Poisson process with non-stationary arrival rate

$$\beta_i(t) := \beta \int_{r_i}^{r_{i+1}} k(x,t)dx, \tag{2.38}$$

proportional to the density of traffic in segment i at time t, where β is the arrival rate of fresh calls per unit traffic mass, and r_i and r_{i+1} are the borders of segment i. Let the call lengths be independent and identically distributed random variables, with common distribution G and mean τ independent of the location and traffic density.

Outage and blocking probabilities

We may distinguish two ways of handling fresh calls that bring the system in a non-feasible state: we may either block and clear the call from the system (*fresh call blocking*), or accept the call in which case the system is said to be in outage (*outage probability*) and (some) calls do not reach their energy per bit to interference threshold ϵ^*, until completion of some (other) call. These 'outage' and 'blocking' cases lead to different stochastic processes recording the number of calls in the segments.

If calls are blocked and cleared when the state is not feasible, the set of feasible states is the finite set $\mathcal{S}$ as defined in (2.35). Let $\{X(t), t \geq 0\}$ be the stochastic process recording the number of non-persistent and moving calls over the segments, which takes values in the finite state space $\mathcal{S}$. A state of the stochastic process is a vector $\mathbf{U} = (n_1, n_2, \cdots, n_{I,} m_{J,} \cdots, m_2, m_1)$, that will be labelled as $\mathbf{U} = (u_1, u_2, \cdots, u_{I,} u_{I+1,} \cdots, u_{I+J})$. When calls are not blocked, but instead all (or some) calls are in outage when the system state is not feasible, then all vectors in the positive orthant

$$\mathcal{S}^{\infty} = \left\{ \mathbf{U} \mid \mathbf{U} = (\mathbf{N}, \mathbf{M}) \in \mathbb{N}^L \right\}, \tag{2.39}$$

are possible system states. Let $\{X^{\infty}(t), t \geq 0\}$ be the corresponding stochastic process.

We are primarily interested in the distribution of calls over the segments $P\left(X^{\infty}(t) = \mathbf{U}\right)$, and $P\left(X(t) = \mathbf{U}\right)$. For the 'outage case' this distribution can be evaluated in closed form:

$$P\left(X^{\infty}(t) = \mathbf{U}\right) = \prod_{s=1}^{L} e^{-\rho_s^{\infty}(t)} \frac{\rho_s^{\infty}(t)^{u_s}}{u_s!}, \tag{2.40}$$

where

$$\rho_s^{\infty}(t) = \tau \lambda_s(t), \tag{2.41}$$

is the time-dependent *load offered* to segment s: the distribution of the number of calls in cell s is Poisson with mean $\rho_s^{\infty}(t)$ proportional to the density of traffic and insensitive to the distribution of the call length G except through its mean τ, see [MW93] for a general framework for networks with unlimited capacity, and [UB01] for a derivation of the insensitivity result (2.40).

For the 'blocking case' the distribution $P\left(X(t) = \mathbf{U}\right)$ cannot be obtained in closed form. However, for the regime of small blocking probabilities, the distribution $P\left(X(t) = \mathbf{U}\right)$ can be adequately approximated using the Modified Offered Load (MOL) approximation:

$$P\left(X(t) = \mathbf{U}\right) \approx P\left(X^{\infty}(t) = \mathbf{U} \mid X^{\infty}(t) \in \mathcal{S}\right) = \frac{\prod_{s=1}^{L} e^{-\rho_s^{\infty}(t)} \frac{\rho_s^{\infty}(t)^{u_s}}{u_s!}}{\sum_{u \in \mathcal{S}} \prod_{s=1}^{L} e^{-\rho_s^{\infty}(t)} \frac{\rho_s^{\infty}(t)^{u_s}}{u_s!}}.$$

The approximation is exact for a loss network in equilibrium. For networks with time-varying rates the MOL approximation is investigated in [MW94] for the Erlang loss queue, and is applied to networks of Erlang loss queues in [AB02]. It is shown that the error of the MOL approximation is decreasing with decreasing blocking probabilities and with decreasing variability of the arrival rate.

Outage and blocking probabilities are now readily obtained. First consider the 'outage case'. As the number of calls in the system increases, all calls suffer a gradual degradation of their QoS. If the energy per bit to interference ratio of a call falls below its target value ϵ^*, then the system is said to be in outage. The outage probability, $P_{out} = P\left(X^{\infty}(t) \notin \mathcal{S}\right)$, is defined as the probability that an (instant) outage occurs to the system. The outage probability of a user in segment j in a cell can be formulated as follows :

$$P_{out} = P\left(\epsilon_j < \epsilon^* \text{ for some j}\right). \tag{2.42}$$

The outage probability cannot be evaluated in closed form due to the complexity of the feasible set $\mathcal{S}$, and will be evaluated via Monte-Carlo simulation.

For the 'blocking case', the fresh call blocking probability must be determined per segment. To this end, define the blocking set of segment k as

$$\mathcal{S}_k = \left\{\mathbf{U} \in \mathcal{S} \mid \ \lambda\left(\mathbf{U} + \mathbf{e}_k\right) > s\right\},$$

where $\mathbf{e}_k$ is the unit vector with entry k equal 1, and all other entries 0. Then, as is shown in [AB02], the blocking probability, $B_k(t)$, of a segment k at time t is approximated as

$$B_k(t) \approx P\left(X^{\infty}(t) \in \mathcal{S}_k \mid X^{\infty}(t) \in \mathcal{S}\right) = \frac{\sum\limits_{\mathbf{U}\in\mathcal{S}_k} \prod\limits_{s=1}^{L} e^{-\rho_s^{\infty}(t)} \frac{\rho_s^{\infty}(t)^{u_s}}{u_s!}}{\sum_{u\in\mathcal{S}} \prod\limits_{s=1}^{L} e^{-\rho_s^{\infty}(t)} \frac{\rho_s^{\infty}(t)^{u_s}}{u_s!}}.$$

The blocking probability cannot be evaluated in closed form due to the complexity of the feasible set $\mathcal{S}$, and will be valuated via Monte-Carlo simulation in the numerical results section.

2.3 Downlink capacity allocation

The assignment of transmission powers to calls is an important problem for network operation, since the interference caused by a call is directly related to the power. In the CDMA downlink, the transmission power is related to the downlink rate. Hence, for an efficient system utilization, it is necessary to adopt a rate allocation scheme in the transmission power assignment.

This section presents a model for system utility optimization based on the feasibility model. In particular, the objective is to find the best border location for both downlink and uplink that maximizes the total number of uplink users and maximizes the total downlink rate.

We model the problem of finding a maximum system utility as a discrete optimization problem. We choose the system utility as the total sum of rates allocated to users. If the rates used are assigned to a certain price, i.e., euro per bit used, then this optimization model can be interpreted as the total revenue of the system. Note that the algorithm we present also works for the other definition of system utility, such as in [TAG02, DNZ03, Jav06, OJB03, XSC01].

2.3.1 Uplink and downlink feasibility

Recall the feasibility condition for downlink and uplink, in (2.17) and (2.33) respectively. Feasibility of power control allocations has been investigated via PF eigenvalues. We are interested in feasibility when the rate and the users distributions are not fixed. Given (2.18) and (2.34), the feasibility conditions in (2.17) and (2.33) can be rewritten as

$$\lambda'(n_i, m_j, r_d) < \frac{2W}{\epsilon_d^*}, \tag{2.43}$$

$$\widehat{\lambda}'(n_i, m_j, r_u) < \frac{2W}{\epsilon_u^*}, \tag{2.44}$$

where

$$\begin{aligned}\lambda'(n_i, m_j, r_d) &= r_d \left(\sum_{i=1}^{I} \alpha n_i + \sum_{j=I+1}^{L} \alpha n_j - 2\alpha \right) \\ &\quad + r_d \sqrt{\left(\sum_{i=1}^{I} \alpha n_i - \sum_{j=I+1}^{L} \alpha n_j \right)^2 + 4 \sum_{i=1}^{I} n_i l_i \sum_{j=I+1}^{L} n_i l_i} \quad ,\end{aligned}$$

and

$$\begin{aligned}\widehat{\lambda}'(n_i, m_j, r_u) &= r_u \left(\sum_{i=1}^{I} n_i + \sum_{j=I+1}^{L} n_j - 2 \right) \\ &\quad + r_u \sqrt{\left(\sum_{i=1}^{I} n_i - \sum_{j=I+1}^{L} n_j \right)^2 + 4 \sum_{i=1}^{I} n_i l_i \sum_{i=I+1}^{L} n_i l_i} \quad .\end{aligned}$$

Equations (2.43) and (2.44) represent the feasibility condition for downlink and uplink where the system parameters W, ϵ_d^* and ϵ_u^* are fixed. Using those expressions, we investigate the relation between user distribution (n_i, n_j), uplink rate r_u and downlink rate r_d. We observe that for $\alpha = 1$, the expression for downlink feasibility and uplink feasibility are the same. Moreover, since the downlink non-orthogonality factor has a value between 0 and 1, i.e., $0 \leq \alpha \leq 1$, for the case of $r_d = r_u = R$ we always have the following relation

$$\lambda'(n_i, m_j, R) \leq \widehat{\lambda}'(n_i, m_j, R). \tag{2.45}$$

This means that the downlink rate can be upgraded while maintaining both uplink and downlink feasibility.

2.3.2 Border optimization

From (2.18) and (2.34), we observe that the PF eigenvalues can be related to the border location. This is done by assigning users from a cell to other cells, i.e., assigning I segments to cell X and $(L - I)$ segments to cell Y, given users distribution $\mathbf{U} = (n_1, n_2, \cdots, n_{I,} n_{I+1,} \cdots, n_{L-1}, n_L)$.

We observe that the downlink PF eigenvalue decreases as the location of the border is located further from the middle of the traffic burst. Therefore, it seems optimal to handle all calls in a single BTS. While in the uplink, the uplink PF eigenvalue decreases as the border is located closer to the middle of the traffic burst. So, from the uplink point of view, it is optimal to equally divide calls over two BTSs. From those two observations, we see that there is a trade-off between uplink and downlink optimal border location. Therefore the border location should be determined by considering both downlink and uplink properties. We formulate an optimization problem to solve the combined downlink-uplink optimal border location in this section.

The arguments above suggest that the optimal downlink rate assignment may be to assign rate zero to all segments except for the segment closest to a BTS. This is clearly not a practical solution. Therefore, in our optimization problem, we add a practical constraint that the number of segments with non-zero rates assignment should be maximized. This means that the rate assignment is fair in the sense that the maximum number of calls is carried with equal rate. The combined optimization problem is formulated as follow:

$$\begin{cases} \text{Find borders locations } I, J \text{ and downlink rate } r_d \text{ that} \\ \text{maximize the system utility and number of carried calls} \\ \text{s.t. uplink feasible \& downlink feasible.} \end{cases} \tag{2.46}$$

In this chapter, the coverage of a cell is equal to the number of segments covered by the cell. Thus, the border of cell X is defined as the point located after segment I and the border of cell Y is defined as the point located before segment J. Using the feasibility conditions expression in (2.43) and (2.44), the problem can be formulated as follow

$$
\begin{array}{ll}
\max\limits_{r_d, I, J} & r_d \left(\sum\limits_{i=1}^{I} n_i + \sum\limits_{j=1}^{J} m_j \right) \\
\text{s.t} & \lambda'(n_i, m_j, r_d) < \frac{2W}{\epsilon_D^*}, \\
& I, J \in \arg\max \left(\sum\limits_{i=1}^{I} n_i + \sum\limits_{j=1}^{J} m_j \right) \\
& \qquad\qquad \text{s.t} \quad \widehat{\lambda}'(n_i, m_j, r_u) < \frac{2W}{\epsilon_U^*}, \\
& \qquad\qquad\qquad i = 1, 2, \cdots, I, \quad j = 1, 2, \cdots, J, \\
& \qquad\qquad\qquad I + J \leq L,
\end{array}
\tag{2.47}
$$

where L is the total number of segments. Note that the constraints are non-convex functions in n_i and m_j. Hence, the optimization problem above is not easy to solve. We propose a decomposition algorithm to solve the optimization problem. From (2.45), we learn that $\left(\sum_{i=1}^{I} n_i + \sum_{j=1}^{J} m_j\right)$ in the objective function is mainly determined by the uplink. Hence to find the optimal solution (I^*, J^*, r_d^*) of the problem above, we construct the following algorithm:

1. First, given the traffic load, we label the number of users in each segment as $U = (u_1, u_2, \cdots, u_k, \cdots, u_L)$, where L is the total number of segments.

2. Next, we assign users for a certain border location. For this purpose, we define an initial border at segment k, $k = 1, 2, \cdots, L$. By putting the initial border at segment k, this means that we assign users in the first k segments to cell X and the next $(L-k)$ segments to cell Y, i.e.,

$$
\left(n_i^k, m_j^k\right) = \begin{cases} n_i = u_i & i = 1, 2, \cdots, k, \\ m_j = u_{L-(j-1)} & j = (k+1), \cdots, L, \end{cases}
$$

   where $\left(n_i^k, m_j^k\right)$ denote the set of assigned users when the initial border located at segment k, $k = 1, 2, \cdots, L$. We denote the initial border as (I_k^0, J_k^0).

3. Next, we check the uplink feasibility given the initial border at segment k, (I_k^0, J_k^0), and the assigned users $\left(n_i^k, m_j^k\right)$, $k = 1, 2, \cdots, L$. We check the uplink feasibility given by the first constraint, i.e.,

$$
\widehat{\lambda}''(n_i^k, m_j^k) < \frac{2W}{\epsilon_u^* r_u}, \tag{2.48}
$$

where $\widehat{\lambda}''(n_i, m_j) = \widehat{\lambda}'(n_i, m_j, r_u) / r_u$, i.e.,

$$\widehat{\lambda}''(n_i^k, m_j^k) = \left(\sum_{i=1}^{I_k^0} n_i^k + \sum_{j=1}^{J_k^0} m_j^k - 2 \right) + \sqrt{\left(\sum_{i=1}^{I_k^0} n_i^k - \sum_{j=1}^{J_k^0} m_j^k \right)^2 + 4 \sum_{i=1}^{I_k^0} p_i n_i^k \sum_{j=1}^{J_k^0} p_j m_j^k} \quad . \tag{2.49}$$

Thus, given the set of users$\left(n_i^k, m_j^k\right)$ and the initial border set (I_k^0, J_k^0), the uplink feasibility is checked as follows

$$\widehat{\lambda}''(n_i^k, m_j^k) = \begin{cases} \text{if} < \frac{2W}{\epsilon_u^*} & \text{then the border is } (I_k^0, J_k^0). \\ \text{if} \geq \frac{2W}{\epsilon_u^*} & \text{then drop segments until feasible.} \end{cases}$$

The dropped segment is the one that contributed the most to $\widehat{\lambda}''(n_i^k, m_j^k)$. From (2.49), we can see that the dropped segment is located close to the cell border. If we drop the segment $J_k = J_k^0$, then we set $I_k' = k$ and $J_k' = (L - k - 1)$. If we drop the segment $I_k = I_k^0$, then we set $I_k' = (k - 1)$ and $J_k' = (L - k)$. Then, we obtain a set of border (I_k, J_k) with a gap of a segment. We repeat those steps until (2.48) is satisfied. Finally, for each k, we obtain a set of border $B_k = (I_k, J_k)$ that supports a maximum number of users, $U_k = \left(n_i^k, m_j^k\right)$, under uplink feasibility constraints.

4. Next, we determine a set of $B_k = (I_k, J_k)$, $k = 1, 2, \cdots, K$, that maximize $\left(\sum_{i=1}^{I_k} n_i^k + \sum_{j=1}^{J_k} m_j^k\right)$. Thus, given the border B_k from step 3, we choose k^* among all k, the set that gives either optimal carried calls,

   $\left(\sum_{i=1}^{I_k} n_i^k + \sum_{j=1}^{J_k} m_j^k\right)$ or optimal carried segments $(I_k + J_k)$.

   Denote the sets of optimal borders determined by the carried call as $O^U = \left\{B_{k1}^U, B_{k2}^U, \cdots, B_{kq}^U\right\}$. Denote the sets of optimal borders determined by the carried segment as $O^S = \left\{B_{k1}, B_{k2}^S, \cdots, B_{kr}^S\right\}$.

5. Given the set border locations that support maximum number of uplink feasible users, i.e., the set O^U and O^S, we determine the maximal downlink rate

$$r_d < \frac{2W}{\epsilon_d^* \left(\lambda''(n_i^k, m_j^k, I_k, J_k)\right)}, \tag{2.50}$$

where

$$\lambda''(n_i, m_j, I, J) = \alpha\left(\sum_{i=1}^{I} n_i + \sum_{j=1}^{J} m_j - 2\right) + \sqrt{\alpha^2\left(\sum_{i=1}^{I} n_i - \sum_{j=1}^{J} m_j\right)^2 + 4\sum_{i=1}^{I} l_i n_i \sum_{j=1}^{J} l_j m_j} \quad .$$

Thus we obtain a set from O^U, i.e.,

$$P^U = \left\{\left(B_{k1}^U, r_{d1}^U\right), \left(B_k^U, r_{d2}^U\right), \cdots, \left(B_k, r_d^U\right)\right\},$$

and a set from U^S,i.e.,

$$P^S = \left\{\left(B_{k1}^S, r_{d1}^S\right), \left(B_{k2}^S, r_{d2}^S\right), \cdots, \left(B_{k2}^S, r_{dr}^S\right)\right\}.$$

6. Finally, we determine the maximal value of $r_d\left(\sum_{i=1}^{I_k} n_i^k + \sum_{j=1}^{J_k} m_j^k\right)$. This is done by checking all k in the sets P^U and P^S.

The above algorithm is numerically illustrated in the next section.

2.4 Numerical results

In this section, we give some numerical examples demonstrating the results of our model. First, we investigate the relation between downlink performance and downlink border location. Second, we consider downlink border optimization under uplink coverage restrictions.

The parameters that are used for this numerical results are those provided in [HT07]: the system chip rate $W = 3.84$ MHz, the required energy per bit to interference ratio $\epsilon^* = 5$ dB, the downlink non orthogonality factor $\alpha = 0.3$, and the path loss exponent $\gamma = 4$. The distance between the two BTSs X and Y is 2000 meter, divided into 40 segments of width 50 meter. We assume that all users use the same uplink rate $r_u = 32$ kbps. For the downlink, we assume that initially all users use the same downlink rate $r_d = 32$ kbps. Additional results for a system with lower rates $r_u = r_d = 12.2$ kbps and lightly loaded non-hot spot cells are provided in [EvdBB05].

2.4.1 Downlink performance

This section investigates the downlink performance, i.e., the outage probability and the blocking probability per segment, for the case of fixed border and moving border. In the first case, we investigate the downlink performance for a moving traffic hot spot for fixed border location. The performance is calculated as a function of the location of the traffic. In the second case, we investigate the downlink optimal border for non-moving traffic. The performance is calculated as a function of the border location. We will investigate results from Monte-Carlo simulation, and a prediction based on the Perron-Frobenius eigenvalue obtained from the offered load in the segments. Sufficient samples are generated to have 95% confidence and 10% relative precision. To facilitate a graphical representation of our results, we will depict blocking probabilities only for those time instances at which the hot spot enters a new segment.

Location of the hot spot: traffic types

Throughout this section, we assume that a block shaped traffic jam of width 10 segments moves from BTS X to BTS Y at constant speed, see Figure 2.2. The load in segments inside the hot spot is 5 Erlang. The location of the hot spot after the $(12+i)$-th segment from BTS X will be referred to as type i traffic, i.e., Figure 2.2 depicts type 1 traffic. In our numerical results we will only consider types $1, \ldots, 5$, as the hot spot location in the area roughly in the middle between the BTSs is most interesting. Notice that type 5 is the mirror image of type 1, with 13 segments between the hot spot and BTS Y.

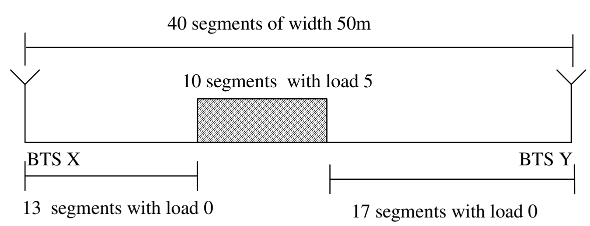


Figure 2.2: *Rectangular hot spot*

First case: fixed border, moving traffic

First consider the commonly studied case of a fixed border located in the middle between the BTSs, i.e., each cell consists of 20 segments. Blocking and outage probabilities can be obtained via Monte-Carlo simulation. Below we will numerically investigate the blocking probabilities per segment for a moving hot spot.

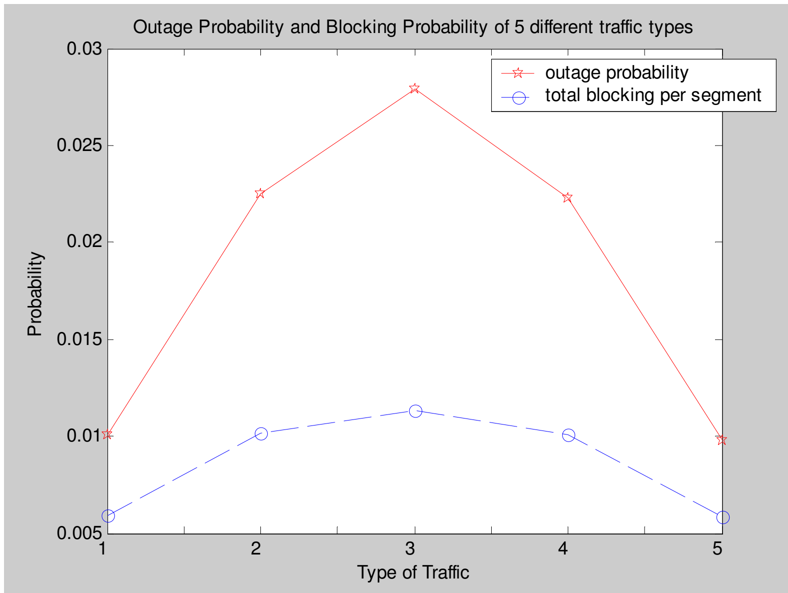


Figure 2.3: *Outage and total blocking for the first case*

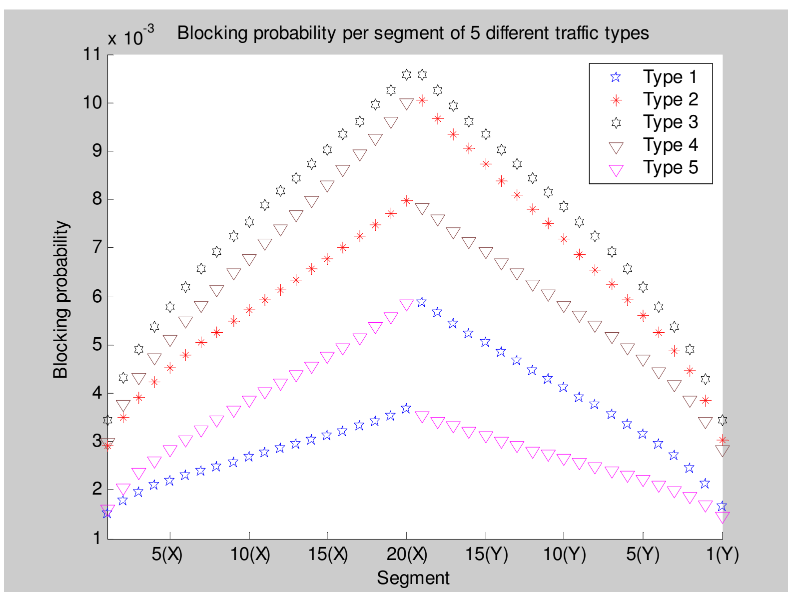


Figure 2.4: *Blocking per segment for the first case*

Figure 2.3 depicts the outage and total blocking probabilities for traffic types $1-5$, where the total blocking is the fraction of blocked fresh calls over the entire area between BTSs X and Y. Both the outage and the total blocking probability do not discriminate between segments. Clearly, type 3 traffic with the hot spot located in the middle between BTSs yields the largest value for the blocking probabilities, in accordance with intuition.

Figure 2.4 depicts blocking probabilities per segment for traffic types $1-5$, that is the fraction of blocked fresh calls counted for each segment separately. As can be seen from the graph, when the hot spot is located more to the left, the blocking probability of the segments in the right is higher (see type 1 and type 2 traffic load) and vice versa. The type 3 case is symmetric. This result shows that as the traffic jam moves closer to the border, the downlink performance gets worse. The result suggests that it is optimal for the downlink to have all calls located in the same cell. This motivates an investigation of the downlink performance when the border is not fixed.

Second case: moving border, non-moving traffic

Let us now investigate the optimal location of the border between the cells for a given traffic pattern, i.e., the location of the border that gives the best downlink performance. For this case, we consider the traffic of type 1.

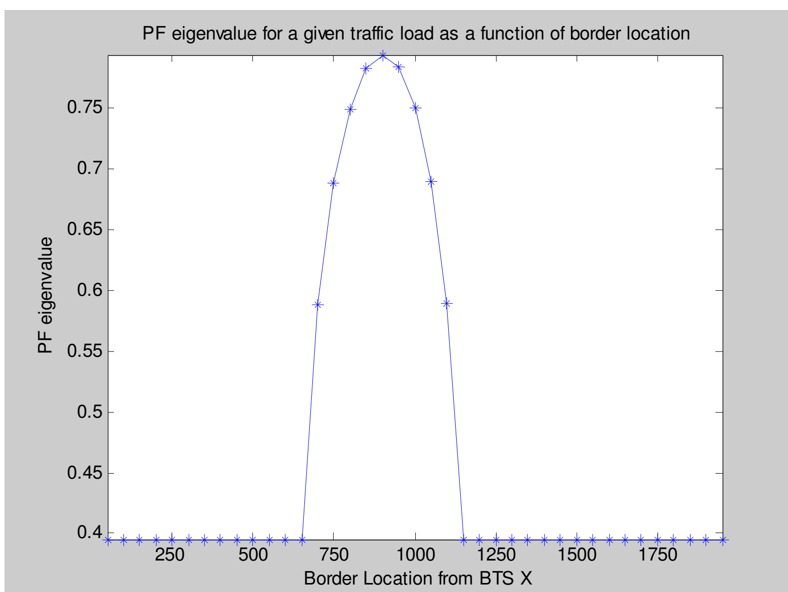


Figure 2.5: *Downlink PF eigenvalue for the second case*

Figure 2.5 depicts the downlink PF eigenvalue λ as a function of the offered load only. The graph has a clear peak for a cell border between roughly 700 and 1100 meters from BTS X. As the feasibility criterion is $\lambda < s$ (recall Theorem 2.2.1 and Eq.(2.17)), from the curve it seems optimal for the cell border to be such that the entire hot spot resides in a single cell. Monte-Carlo simulation of the blocking probabilities per segment for type 1 traffic and different locations of the border at 700, 900, 1000 and meters from BTS X as depicted in Figure 2.6 support this observation: congestion in the downlink can be reduced by allocating the entire traffic burst into one cell.

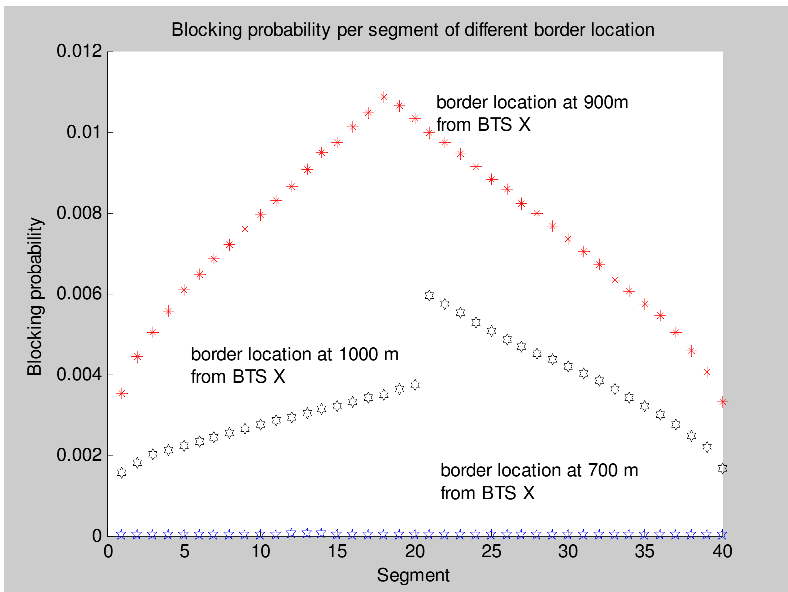


Figure 2.6: *Blocking per segment for the second case*

The numerical results, which based on the downlink only, showed that congestion in the downlink can be reduced by allocating the entire traffic burst into one cell. This in clear contrast with the well-known uplink result that indicates that the load should be evenly divided over the cells. Thus, there is a trade-off between downlink congestion and uplink congestion: the location of the border should be determined by considering both uplink and downlink. This problem will be addressed in the next section.

2.4.2 Downlink rate optimization

Next, we give some numerical examples demonstrating the results of our model. First, we investigate the relation between downlink performance and downlink border location. Second, we consider downlink border optimization under uplink coverage restrictions.

Optimal border

This section investigates the optimal border location based on the optimization problem of Eq.(2.47). In the first case, we fix the traffic load to be of type 1 as in Figure 2.2. This algorithm (in step 1-3) first investigates the possible border locations that give the optimal number of carried calls or carried segments.

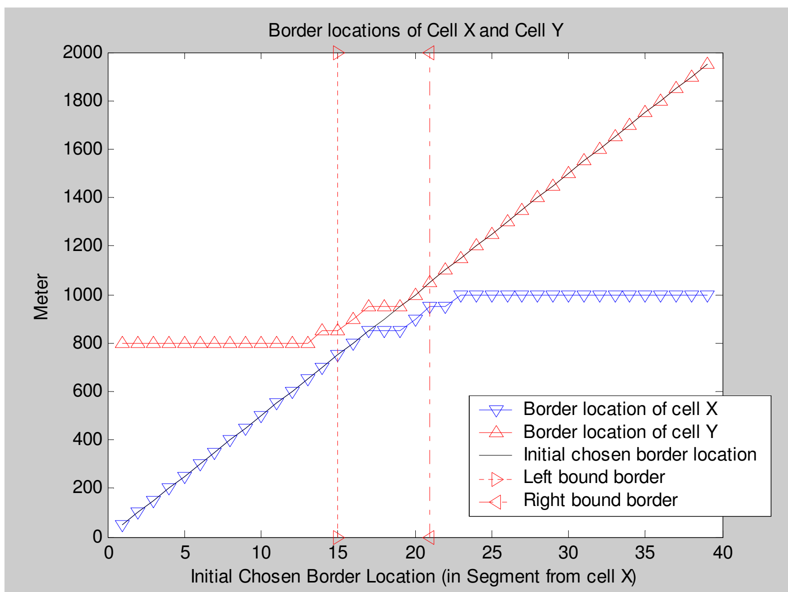


Figure 2.7: *Optimal border location*

Figure 2.7 depicts the optimal border locations, $B_k = (I_k^*, J_k^*)$, as a function of the initial border location placed at segment k. Thus, the optimal cell borders (step 4) are $O^U = O^S = \{(U_{15}, B_{15}), (U_{16}, B_{16}), \cdots, (U_{22}, B_{22})\}$ obtained for k between 15 and 22, as indicated by the vertical lines in Figure 2.7. Notice that there is coverage gap in the middle between BTS X and Y.

Given the optimal set of carried segments O^S, step 5 of the algorithm determines the possible upgrade of downlink rate r_d using Eq.(2.50). Figure 2.8 depicts the utility function $r_d\left(\sum_{i=1}^{I_k} n_i^k + \sum_{j=1}^{J_k} m_j^k\right)$ as a function of k.

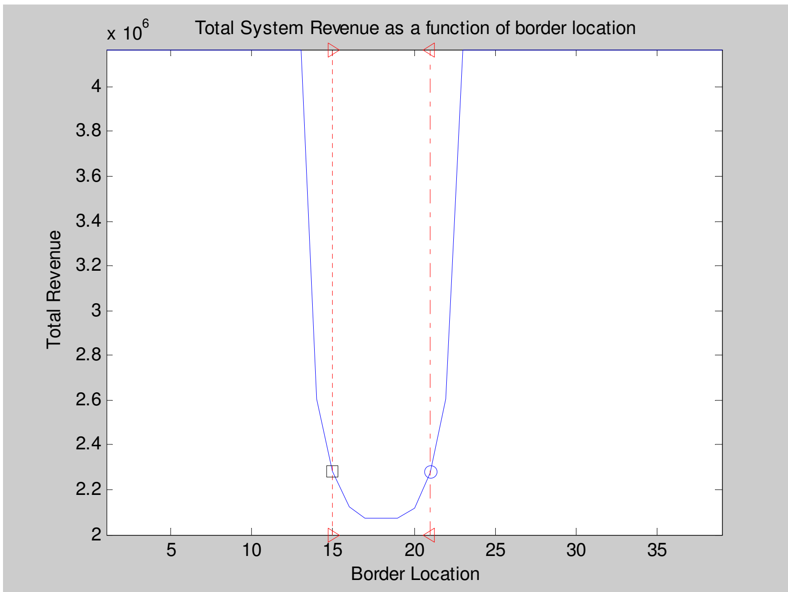


Figure 2.8: *Perron-Frobenius eigenvalue*

The maximal utility value is denoted as star in Figure 2.8. Thus, the border location that gives maximal utility is at $k = 22$. The optimal border location for cell X is at $I = 21$ (950m from BTS X) and the optimal border location for cell Y is at $J = 18$ (950m from BTS Y). The maximum is obtained with the border located further from the center of the hot spot/traffic burst: maximal system utility is obtained by putting the borders such that most of the traffic is covered in a single cell.

Notice from Figure 2.8 that the system utility can be increased when we let the system support less carried calls, in this case the per call downlink rate is higher, but the number of carried calls is lower. This shows the fairness trade-off between number of carried calls and the system utility: by serving less calls the remaining calls would be able to achieve higher total utility. A similar result is found in [Sir02], where the uplink is investigated, only.

Next, we investigate the optimal border location for the case of moving traffic. The objective is to understand the optimal border location and its optimal system utility. For this purpose, we let the hot spot of type 1 (see Figure 2.2) moves from BTS X to BTS Y. In this example, we consider only 5 steps. For each step, we investigate the optimal border location that gives maximal utility. Figure 2.9 depicts the optimal border locations in each step that gives the maximum system utility and illustrates that in this numerical example there is no distinction made in our algorithm using the optimal number of carried calls and the optimal number of carried segments.

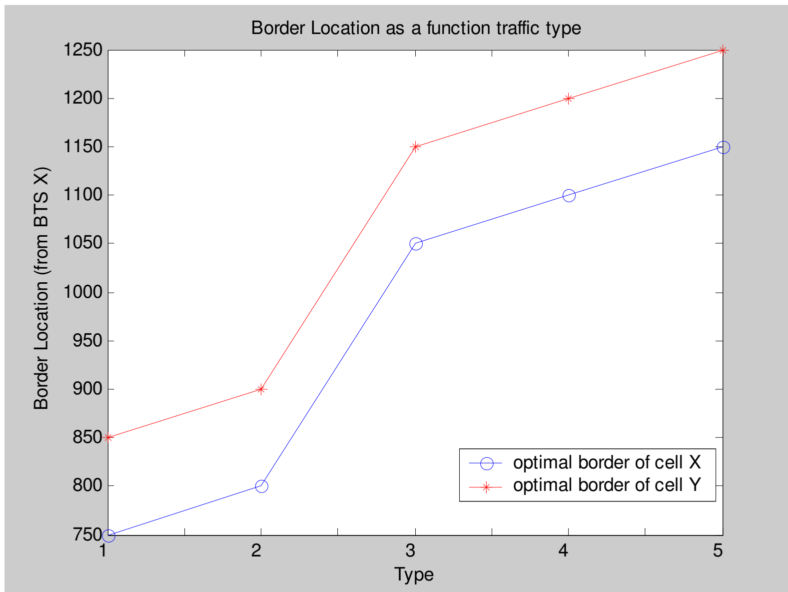


Figure 2.9: *Optimal border location*

Figure 2.10 depicts that the maximum system utility based on carried calls and the the maximum system utility based on carried segments. As the traffic burst moves closer to the middle of the cell, the optimal total revenue decreases. Furthermore, Figure 2.10 indicates that it is optimal to choose the border location such that most of the traffic burst is covered by a single cell.

Non-persistent calls

Now, we investigate the optimal border location for the case of non-persistent calls by Monte-Carlo simulation for traffic type 1 (see Figure 2.2). For each realization, we perform the algorithm in Section 2.3.2

Figure 2.11 depicts the probability that we obtain a location of the border in a particular place. The figure shows that there are two peaks for the border of cell X, i.e., at 650m and at 1000m, and by symmetry also two peaks for the border

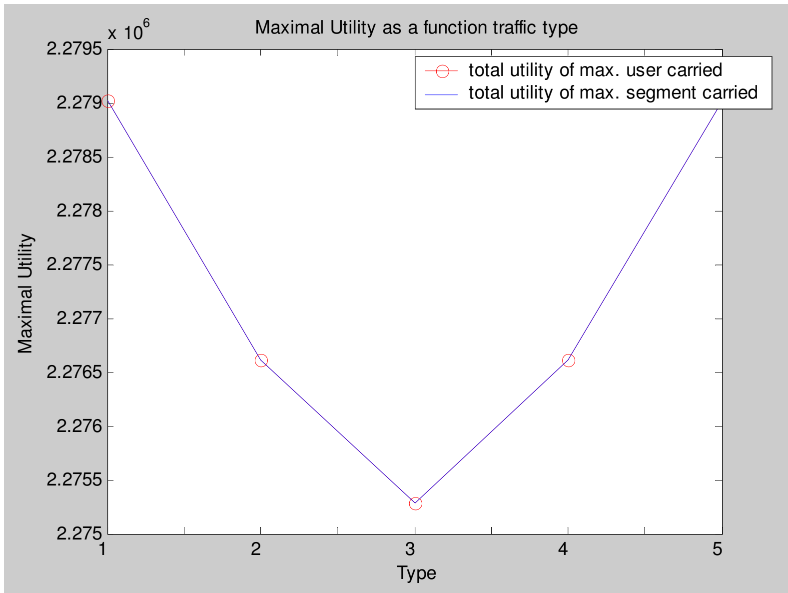


Figure 2.10: *Optimal system utility*

of cell Y, i.e., at 800m and at 1150m. The peak at 650m for cell X is dominant. This is in contrast with the result for persistent calls. The discrepancy is due to the algorithm, that starts including calls in cell X from the left. In the case of a tie in revenue, it chooses the left border thus favouring the left border location. These results show that also in the case of non-persistent calls the optimal border location includes most of the traffic burst in a single cell, i.e., either in cell X or in cell Y.

This is more clearly visible in Figure 2.12 that depicts the optimal border location for symmetric traffic (Type 3), i.e., in the setting of non-persistent calls when two boundary locations around 750m and 1250m yield the same revenue, the algorithm selects the boundary at 750m.

From those two examples, we can conclude that the optimal system revenue, i.e., with maximal number of uplink users and maximal downlink rate, is obtained by covering most of the traffic in a single cell.

Thus, by taking the uplink that determines coverage into account, we have developed a downlink rate optimization algorithm and have investigated the optimal cell border based on both uplink and downlink interference. The results indicate that the optimal border location that maximizes the system utility (downlink rate) can be obtained by including most of the carried traffic into a single cell.

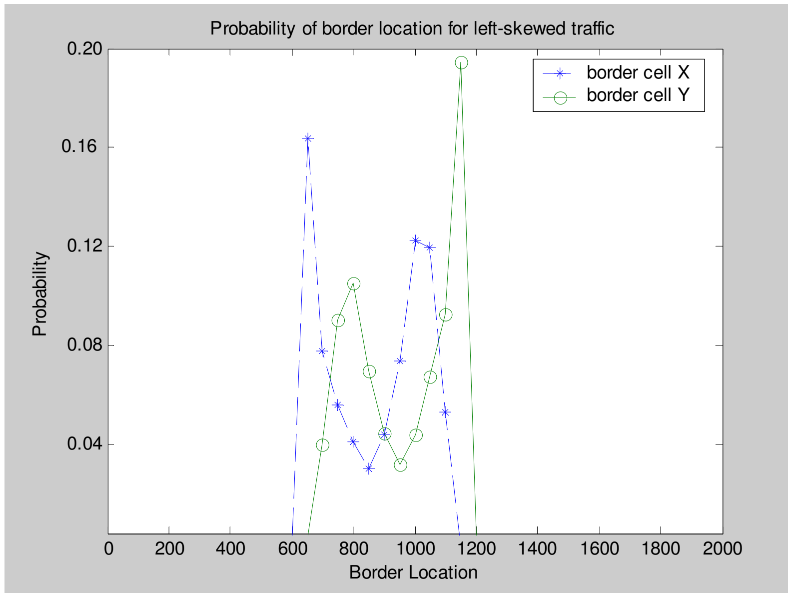


Figure 2.11: *Simulated border location for left-skewed traffic*

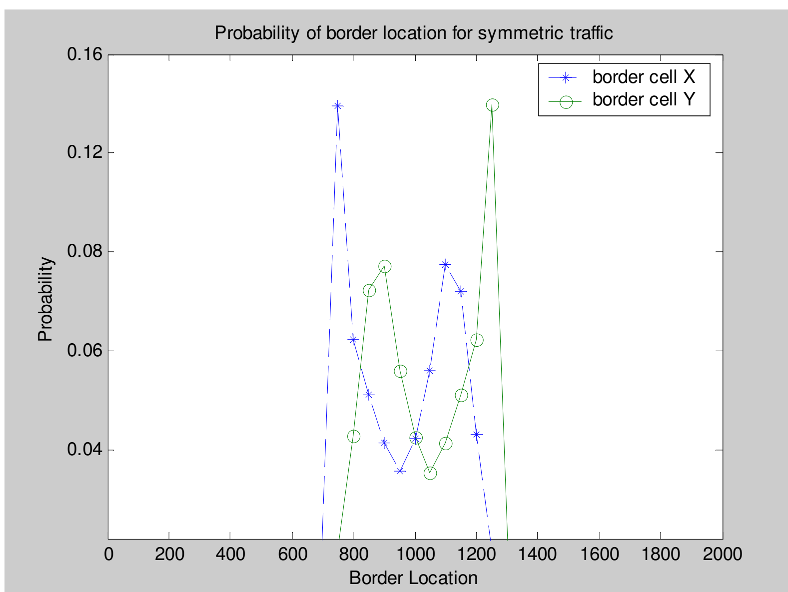


Figure 2.12: *Simulated border location for symmetric traffic.*

2.5 Conclusions

This chapter has provided a model for characterizing downlink and uplink power assignment feasibility. We have obtained an explicit decomposition of system and user characteristics, and have provided an explicit analytical expression for the Perron-Frobenius eigenvalue that determines feasibility and blocking probabilities. Based on this result we have numerically investigated blocking probabilities and found for the downlink that it is best to allocate all calls to a single cell.

Moreover this chapter has also provided a model for determining an optimal cell border in CDMA networks. We have formulated a joint uplink and downlink optimization problem for the downlink and uplink power assignment feasibility. Based on the Perron-Frobenius eigenvalue of the power assignment matrix, we have reduced the downlink rate allocation problem to a set of multiple-choice knapsack problems, yielding an approximation of the downlink rate allocation. We used our combined downlink and uplink feasibility model to determine cell borders for which the system throughput, expressed in terms of downlink rates, is maximized.

This approach proves to have several advantages. First, the discrete optimization approach has eliminated the rounding errors due to continuity assumptions of the downlink rates. Using our model, the exact rate that should be allocated to each user can be indicated. Second, the rate allocation approximation we have proposed guarantees that the solution obtained is close to the optimum. Moreover, we have control on the error of the approximation and the running time of the algorithm. Last, the result of our method confirms the intuitive rate allocation in CDMA systems, i.e., users with lower interference obtain maximum rate. The numerical results have shown that the system utility is maximized when other-cell interferences are minimized. Therefore, users close to the border may receive 0 rate. Such a rate allocation may seem unfair. In the next chapter, we discuss more general situation of downlink rate allocation.

Chapter 3

A Combinatorial Approximation of Two-Cell Downlink Rate Allocation

3.1 Introduction

The assignment of transmission powers to calls is an important problem for network operation, since the interference caused by a call is directly related to the power. In the CDMA downlink, the transmission power is related to the downlink rates. Hence, for an efficient system utilization, it is necessary to adopt a rate allocation scheme in the transmission powers assignment.

In this chapter we propose a rate and power allocation scheme for obtaining a close to optimum throughput for the downlink in a Universal Mobile Telecommunication System (UMTS) located on a highway. In accordance with the UMTS standard, the rates are chosen from a discrete set. Our goal is to assign rates to users, such that the utility of the system is maximized. We measure the satisfaction of a user in segment $i, i \in \{1, ..., L\}$ by means of a positive utility function $u_i(R_i)$. For a presentation of the utility functions commonly used in the literature see [TAG02]. The utility functions describing the satisfaction of the users have a very general form and do not have to satisfy any convexity requirement. Thus, our goal is to allocate rates from a discrete and finite set R=$\{R_1, ..., R_K\}$ to the users such that the total utility, i.e., the sum of the utilities of all users, is maximized under the condition that the prescribed quality of service is met for all users and that a feasible power assignment exists.

For modeling the network, we use the model proposed in [EvdBB05], which enables a characterization of downlink power feasibility via the Perron-Frobenius

(PF) eigenvalue of a suitably chosen matrix. Moreover, based on the explicit analytical expression of the PF eigenvalue reduces the rate optimization problem to a series of multiple choice knapsack problems, that can be solved efficiently by standard combinatorial optimization techniques. Thus the rate allocation problem is NP-hard, so it is very unlikely that polynomial time algorithms exist (unless P=NP). The algorithm we design is actually a Fully Polynomial Time Approximation Scheme (FPTAS) for the rate optimization problem. The main advantages of this approach are that, by considering discrete rates, we avoid the rounding errors due to continuity assumptions and that, given an error bound ϵ, we can find a solution of value at least $(1 - \epsilon)$ times the optimum in polynomial time in the size of the input data and $(1/\epsilon)$. Moreover, the algorithm can be applied for a very large family of utility functions. Furthermore, our results indicate that the optimization problems for different cells are loosely coupled by a single interference parameter. If this parameter were known, the optimization problems for each cell could be independently solved.

In particular, we develop a joint uplink and downlink optimization model with downlink rate differentiation. There are two objectives of our model. The first objective is to find a set of possible border location that maximizes the total number of uplink users. Then, given the set of border locations, we find an approximation of downlink rates allocation that maximizes the total sum of downlink rates allocated

3.2 Downlink rate differentiation

The downlink rate assignment problem has been extensively studied in the literature [Ber01, DNZ02, Jav06, OJB03, Sir02]. In [DNZ02], Duan et al. present a procedure for finding the power and rate allocations that minimizes the total transmit power in one cell. In [Jav06], Javidi analyzes several rate assignments in the context of the trade-off between fairness and overall throughput. The rates are supposed to be continuous and the algorithms proposed for the rate allocation are based on solving the Lagrangean dual. Another approach for joint optimal rates and powers allocation, based on Perron-Frobenius theory, is proposed by Berggren [Ber01] and by O'Neill et al. [OJB03]. Berggren [Ber01] describes a distributed algorithm for assigning base station transmitter (BTSs) powers such that the common rate of the users is maximized, while in [OJB03] multiple rates are considered. Again, both algorithms assume continuous rates. In [EvdBB05], Endrayanto et al. present a model for characterizing downlink and uplink power assignment feasibility, for a single data rate. Boucherie et al. [BBEW06] extended this model to two cells. They propose a downlink rate allocation scheme which approximates very close the maximum of a generally chosen utility function. We generalize the single rate model assumption in Chapter 2 into different rates per segment where r_i is one of the rates from a given finite set by the system (as in

[HT07]). Then Eq.(2.6) becomes

$$V(r_i) = \frac{\epsilon_d^* r_i}{W + \alpha \epsilon_d^* r_i}, \text{ for } i \in \{1, ..., L\}. \tag{3.1}$$

Thus, the system (2.9) will change into

$$\begin{cases} P_i = \alpha V(r_i) \sum_{j=1}^{I} P_j n_j + V(r_i) l_i \sum_{j=I+1}^{L} P_j n_j + V(r_i) l_{i,X}^{-1} N_0, \\ \qquad \text{for } i \in 1, ..., I, \\ P_i = V(r_i) l_i \sum_{j=1}^{I} P_j n_j + \alpha V(r_i) \sum_{j=I+1}^{L} P_j n_j + V(r_i) l_{i,Y}^{-1} N_0, \\ \qquad \text{for } i \in I+1, ..., L, \\ P_i \geq 0, \text{ for } i \in 1, ..., L. \end{cases} \tag{3.2}$$

Without loss of generality, in this chapter we assume that all users in the same segment have the same rate r_i chosen from a finite set of possible transmission rates $r_i \in \{R_1, ..., R_K\}$. Note that, if in a segment the maximum rate r_K is not requested, then $r_i \in \{R_1, ..., R_{K-1}\}$. This assumption leads to a better use of the resources. Let $\mathbf{R}_X = (r_1, r_2, \cdots, r_I)$, respectively $\mathbf{R}_Y = \left(r_1, r_2, \cdots, r_{(L-I)}\right)$, be the rates assigned to users in cell X, respectively cell Y.

3.2.1 Dimension reduction

Next we show that the feasibility of (3.2) is equivalent to the feasibility of a system with 2 equations (each of them characterizing one cell) and a positivity constraint.

Lemma 3.2.1 *System (3.2) is feasible if and only if the following system is feasible:*

$$\begin{cases} \left(1 - \alpha \sum_{j=1}^{I} V(r_i) n_i\right) x - \sum_{j=1}^{I} V(r_j) n_j l_j y = \sum_{j=1}^{I} V(r_j) n_j l_{j,X}^{-1} N_0, \\ - \sum_{j=I+1}^{L} V(r_j) n_j l_j x + \left(1 - \sum_{j=I+1}^{L} V(r_j) n_j\right) y = \sum_{j=I+1}^{L} V(r_j) n_j l_{j,Y}^{-1} N_0, \\ x \geq 0, y \geq .0 \end{cases} \tag{3.3}$$

Proof. ($\Longrightarrow$)Let $\mathbf{P}$ be a positive solution of (3.2). In system (3.2) multiply each equation with the number of users in the corresponding segment and then add the first I equations and then the other $(L-I)$. It follows that
$(x, y) = (\sum_{i=1}^{I} n_i P_i, \sum_{i=I+1}^{L} n_i P_i)$ verifies (3.3).

($\Longleftarrow$)Let (x, y) be a solution of (3.3). Define:

$$P_i = \begin{cases} \alpha V(r_i)x + V(r_i)l_i y + V(r_i)l_{i,X}^{-1} N_0, & \text{for } i \in \{1, ..., I\}, \\ V(r_i)l_i x + \alpha V(r_i)y + V(r_i)l_{i,Y}^{-1} N_0, & \text{for } i \in \{I+1, ..., L\}. \end{cases} \tag{3.4}$$

By simple substitution in (3.2) it can be shown that $\mathbf{P}$ is a solution of (3.2). ■

Lemma 3.2.1 reduces the number of calculations involved in characterizing the power feasibility, since it is straightforward to verify that a system with 2 equations in 2 positive variables is feasible.

System (3.3) can be rewritten in the following form:

$$(\mathbf{I} - \mathbf{T}) \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{c}, \tag{3.5}$$

where

$$\mathbf{T} = \begin{pmatrix} \alpha \sum_{i=1}^{I} V(r_i)n_i & \sum_{i=1}^{I} V(r_i)n_i l_i \\ \sum_{i=I+1}^{L} V(r_i)n_i l_i & \alpha \sum_{i=I+1}^{L} V(r_i)n_i \end{pmatrix},$$

$$\mathbf{c} = N_0 \begin{pmatrix} \sum_{i=1}^{I} V(r_i)n_i l_{i,X}^{-1} \\ \sum_{i=I+1}^{L} V(r_i)n_i l_{i,Y}^{-1} \end{pmatrix}.$$

Hence, as it is a system of two by two, the explicit expression of the PF eigenvalue of $\mathbf{T}$ with rate differentiation can be calculated easily:

$$\begin{aligned} \lambda(\mathbf{T}) = \frac{1}{2} \left(\sum_{i=1}^{I} \alpha V(r_i)n_i + \sum_{i=I+1}^{L} V(r_i)n_i \right) \\ + \frac{1}{2} \sqrt{ \left(\sum_{i=1}^{I} \alpha V(r_i)n_i - \sum_{i=I+1}^{L} \alpha V(r_i)n_i \right)^2 + 4 \left(\sum_{i=1}^{I} V(r_i)n_i l_i \right) \left(\sum_{i=I+1}^{L} V(r_i)n_i l_i \right) }. \end{aligned} \tag{3.6}$$

3.2.2 Cell decomposition

The next theorem gives another motivation for discretization.

Theorem 3.2.1 *For a given rate allocation r, a feasible power allocation exists, i.e., system (3.2) is feasible, if and only if*

$$\begin{cases} \sum_{i=1}^{I} \alpha V(r_i) n_i < 1, \\ \sum_{i=I+1}^{L} \alpha V(r_i) n_i < 1, \\ (1 - \sum_{i=1}^{I} \alpha V(r_i) n_i)(1 - \sum_{i=I+1}^{L} \alpha V(r_i) n_i) > \left(\sum_{i=1}^{I} V(r_i) n_i l_i \right) \left(\sum_{i=I+1}^{L} V_i n_i l_i \right). \end{cases}$$

Proof. ($\Longrightarrow$)From Equation (2.17), we know that the system is feasible if and only if $\lambda(\mathbf{T}) < 1$. Given Equation (3.6), the expression $\lambda(\mathbf{T}) < 1$ is equivalent with the following system:

$$\begin{cases} \sum_{i=1}^{I} \alpha V(r_i) n_i + \sum_{i=I+1}^{L} \alpha V(r_i) n_i < 2, \\ (1 - \sum_{i=1}^{I} \alpha V(r_i) n_i)(1 - \sum_{i=I+1}^{L} \alpha V(r_i) n_i) > \left(\sum_{i=1}^{I} V(r_i) n_i l_i \right) \left(\sum_{i=I+1}^{L} V(r_i) n_i l_i \right). \end{cases} \tag{3.7}$$

Since $\sum_{i=1}^{I} \alpha V(r_i) n_i$ and $\sum_{i=I+1}^{L} \alpha V(r_i) n_i$ cannot be both larger than 1 without violating the first inequality of (3.7), it follows that the system (3.7) verifies (3.2.1).

($\Longleftarrow$) From the first two equations of (3.2.1), we have

$$\sum_{i=1}^{I} \alpha V(r_i) n_i < 1, \tag{3.8}$$

and

$$\sum_{i=I+1}^{L} \alpha V(r_i) n_i < 1. \tag{3.9}$$

Then, it follows directly

$$\sum_{i=1}^{I} \alpha V(r_i) n_i + \sum_{i=I+1}^{L} \alpha V(r_i) n_i < 2. \tag{3.10}$$

Then the system in Theorem (3.2.1) follows directly from the system (3.7). ■

Theorem 3.2.1 provides a clear motivation for discretizing the cells into segments, since it facilitates obtaining an analytical model for characterizing the transmit power feasibility for a certain rate allocation and a certain user distribution. Moreover, we observe that the first two conditions we obtained characterize the two cells separately and the third contains products of factors depending only of one cell.

3.3 The rate optimization problem

Let $R = \{R_1, R_2, ..., R_K\}$ be the set of admissible rates, where $R_1 < R_2 < ... < R_K$. The decision of dropping the users of a segment is equivalent with assigning zero rate to the respective segment. Thus, we assume that the minimum rate is $R_1 = 0$.
The problem of allocating rates from the set R to users such that the total utility of the users is maximized, under the condition of ensuring the required Quality of Service and a feasible power assignment, can be formulated as follows:

$$
(P) \qquad \begin{array}{ll}
\max & \displaystyle\sum_{i=1}^{L} u_i(r_i) \\
\text{s.t.} & \left(\dfrac{E_b}{I_0}\right)_i (r,p) = \epsilon_D^*, \text{ for each user in segment } i, \\
 & r_i \in \{R_1, ..., R_K\}, \text{ for each } i \in \{1, ..., L\}, \\
 & p_i \geq 0 \text{ for each } i \in \{1, ..., L\},
\end{array}
$$

where r_i, respectively p_i represent the rate, respectively the power allocated to segment i and ϵ_D^* is the threshold for the the energy per bit to interference ratio.

We are interested in designing an algorithm for assigning rates to segments in such a way that a throughput of at least $(1 - \epsilon)$ times the optimum is obtained, in a time polynomial in the size of an instance and $\frac{1}{\epsilon}$. Such an algorithm would be a fully polynomial approximation scheme (FPTAS) for problem (P). We distinguish three main steps in the design of the algorithm:

1. First we show that finding an optimal solution of (P) can be reduced to solving a set of optimization problems $\{P_1(t), P_2(t) | t \in [t_{min}, t_{max}]\}$, where $P_1(t)$ characterize the first cell, $P_2(t)$ characterize the second cell and the interval $[t_{min}, t_{max}]$ is an interval depending on the system and the user distribution.

2. Then we show that $P_1(t)$, respectively $P_2(t)$ are multiple choice knapsack problems, for which efficient algorithms are known.

3. Finally, we will prove that to find a solution of value at least $(1 - \epsilon)$ times the optimum, for an $\epsilon > 0$, we only have to solve $P_1(t)$ and $P_2(t)$ for $O(\frac{1}{\epsilon})$, $t \in [t_{min}, t_{max}]$. Since to solve $P_1(t)$, respectively $P_2(t)$ we can apply known FPTAS (see e.g. [CHW76]) for the multiple choice knapsack problem, the algorithm we propose is a FPTAS for (P).

Multiple-choice knapsack formulation

We proceed with the first step of the analysis. Theorem 3.2.1 implies that the optimization problem (P) is equivalent with the following problem:

$$(P') \quad \begin{aligned} \max \quad & \sum_{i=1}^{L} u_i(r_i), \\ \text{s.t.} \quad & \sum_{i=1}^{I} \alpha V(r_i) n_i < 1, \\ & \sum_{i=I+1}^{L} \alpha V(r_i) n_i < 1, \\ & \left(1 - \sum_{i=1}^{I} \alpha V(r_i) n_i\right)\left(1 - \sum_{i=I+1}^{L} \alpha V(r_i) n_i)\right) > H_X H_Y, \\ & r_i \in \{R_1, ..., R_K\}, i \in \{1, ..., L\}. \end{aligned}$$

where $H_X = \left(\sum_{i=1}^{I} V(r_i) n_i l_i\right)$ and $H_Y = \left(\sum_{i=I+1}^{L} V_i(r_i) n_i l_i\right)$.

Note that if the rate assignment in one of the cells is known, the problem of assigning rates to the segments of the other cell reduces to a multiple choice knapsack problem. The multiple choice knapsack problem is a NP-hard problem, for which a FPTAS based on dynamical programming is proposed in [CHW76]. In a multiple choice knapsack problem the following data are given: the sizes and the profits of a set of objects, which are divided into disjoint classes, and the volume of a knapsack. The goal is to choose the set of objects with maximum profit among the sets of objects that fit into the knapsack and contain one object from each class. If, for example, the rates in the cell assigned to BTS Y were known, then, based on (P'), the problem of allocating rates to the segments in the cell assigned to BTS X becomes:

$$\begin{aligned} \max \quad & \sum_{i=1}^{L} u_i(r_i) \\ \text{s.t.} \quad & \sum_{i=1}^{I} V(r_i) n_i (\alpha + l_i \frac{\sum_{i=I+1}^{L} \alpha V(r_i) n_i l_i}{1 - \sum_{i=I+1}^{L} \alpha V(r_i) n_i}) < 1, \\ & r_i \in \{R_1, ..., R_K\}, \text{ for each } i \in \{1, ..., I\}. \end{aligned}$$

This is a multiple choice knapsack problem with the following data: the objects are the pairs $\{(i, s), i \in \{1, ..., I\}, s \in \{1, ..., K\}\}$, a class consists of the objects corresponding to the same segment, the profit of an object (i, s) is $u_i(R_s)$ and its size is $V(R_s) n_i (\alpha + l_i \frac{\sum_{i=I+1}^{L} \alpha V(r_i) n_i l_i}{1 - \sum_{i=I+1}^{L} \alpha V(r_i) n_i})$. The volume of the knapsack is 1.

Hence, if we knew the rate allocation in one of the cells, we could find a rate allocation for the segments in the other cell by applying an algorithm for the multiple choice knapsack problem. Since this also holds for the case where all the segments in one cell receive zero rate, in the following we may assume that in cell X there is at least one segment which receives non-zero rate.

Under these assumptions, problem (P') can be rewritten as:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{L} u_i(r_i) \\
\text{s.t.} \quad & \sum_{i=1}^{I} \alpha V(r_i) n_i < 1, && (3.11) \\
& \sum_{i=I+1}^{L} \alpha V(r_i) n_i < 1, \\
& \frac{\left(1 - \sum_{i=1}^{I} \alpha V(r_i) n_i\right)}{\left(\sum_{i=1}^{I} V(r_i) n_i l_i\right)} > \frac{\left(\sum_{i=I+1}^{L} V_i(r_i) n_i l_i\right)}{\left(1 - \sum_{i=I+1}^{L} \alpha V(r_i) n_i\right)}, && (3.12) \\
& \sum_{i=1}^{I} r_i > 0, && (3.13) \\
& r_i \in \{R_1, ..., R_K\}, \text{ for each } i \in \{1, ..., L\}.
\end{aligned}
$$

Constraint (3.13) ensures that at least one segment in cell X will receive non zero rate. Remark that the variables and parameters characterizing the two cells are well separated in (P'). This suggests a decomposition of (P') into a set of problems corresponding to the first cell and one corresponding to the second cell. Denote

$$
\begin{aligned}
t_{min} &= \min_{r \in R^L} \frac{\sum_{i=I+1}^{L} V_i(r_i) n_i l_i}{1 - \sum_{i=I+1}^{L} \alpha V(r_i) n_i}, \\
t_{max} &= \max_{r \in R^L, r \neq 0} \frac{1 - \sum_{i=1}^{I} \alpha V(r_i) n_i}{\sum_{i=1}^{I} V(r_i) n_i l_i}.
\end{aligned}
$$

From (3.11)-(3.13) follows that (P') is feasible if and only if $\alpha V(R_1) \min_{i \in \{I+1,...,L\}} n_i l_i < 1$ and $t_{min} \leq t_{max}$. In what follows, we suppose that these two conditions are always satisfied.

For each $t \in [t_{min}, t_{max}]$ consider the following problems, $(P_1(t))\,, (P_2(t))$:

$$
(P_1(t)) \qquad \begin{array}{ll}
\max & \sum_{i=1}^{I} u_i(r_i) \\
\text{s.t.} & \dfrac{1 - \sum_{i=1}^{I} \alpha V(r_i) n_i}{\sum_{i=1}^{I} V(r_i) n_i l_i} > t, \\
& \sum_{i=1}^{I} r_i > 0, \\
& r_i \in \{R_1, ..., R_K\}, \text{ for each } i \in \{1, ..., I\},
\end{array}
$$

and

$$
(P_2(t)) \qquad \begin{array}{ll}
\max & \sum_{i=1}^{L} u_i(r_i) \\
\text{s.t.} & t > \dfrac{\sum_{i=I+1}^{L} V_i(r_i) n_i l_i}{1 - \sum_{i=I+1}^{L} \alpha V(r_i) n_i}, \\
& r_i \in \{R_1, ..., R_K\}, \text{ for each } i \in \{I+1, ..., L\}.
\end{array}
$$

Let OPT denote the optimal value of the optimization problem (P') and $OPT_1(t)$, respectively $OPT_2(t)$, be the optimal values of $(P_1(t))$, respectively $(P_2(t))$. In the following lemma we prove that we can find OPT by solving $(P_1(t))$ and $(P_2(t))$ for all $t \in [t_{min}, t_{max}]$.

Lemma 3.3.1 $OPT = \underset{t \in [t_{min}, t_{max}]}{max} OPT_1(t) + OPT_2(t)$.

Proof. Consider a $t \in [t_{min}, t_{max}]$. Let $(\bar{r}_1, ..., \bar{r}_I)$, respectively $(\tilde{r}_{I+1}, ..., \tilde{r}_L)$, be optimal solutions of $(P_1(t))$, respectively $(P_2(t))$. Clearly, $(\bar{r}_1, ..., \bar{r}_I, \tilde{r}_{I+1}, ..., \tilde{r}_L)$ is a feasible solution of (P'), and therefore $OPT_1(t) + OPT_2(t) \leq OPT$. We proved that $\max_{t \in [t_{min}, t_{max}]} OPT_1(t) + OPT_2(t) \leq OPT$. In order to prove the reverse inequality, consider an optimal solution r^* of (P). Let

$$
t = \frac{1 - \alpha \sum_{i=1}^{I} V(r_i^*) n_i}{\sum_{i=1}^{I} V(r_i^*) n_i p_i}.
$$

Since $(r_1^*, ..., r_I^*)$ is feasible for $(P_1(t))$ and $(r_{I+1}^*, ..., r_L^*)$ is feasible for $(P_2(t))$, $OPT \leq OPT_1(t) + OPT_2(t)$. ■

Lemma 3.3.1 implies that an optimal rate allocation can be found by solving independently the set of optimization problems $\{P_1(t)|t \in [t_{min}, t_{max}]\}$ and $\{P_2(t)|t \in [t_{min}, t_{max}]\}$ where each set characterizes only one cell, the cells interacting only through the parameter t.

Solving the multiple-choice knapsack problem

Next we show that $(P_1(t))$ and $(P_2(t)))$ are multiple choice knapsack problems, which can be efficiently solved. For this, we rewrite $(P_1(t))$ and $(P_2(t)))$ in the following form:

$$
(P_1(t)) \qquad \begin{aligned}
\max \quad & \sum_{i=1}^{I} u_i(r_i) \\
\text{s.t.} \quad & \sum_{i=1}^{I} V(r_i) n_i (\alpha + l_i t) < 1, \\
& \sum_{i=1}^{I} r_i > 0, \\
& r_i \in \{R_1, ..., R_K\}, \text{ for each } i \in \{1, ..., I\},
\end{aligned}
$$

and

$$
(P_2(t)) \qquad \begin{aligned}
\max \quad & \sum_{i=1}^{L} u_i(r_i) \\
\text{s.t.} \quad & \sum_{i=I+1}^{L} V(r_i) n_i (\alpha t + l_i) < t, \\
& r_i \in \{R_1, ..., R_K\}, \text{ for each } i \in \{I+1, ..., L\}.
\end{aligned}
$$

The input to the multiple choice knapsack problems $(P_1(t))$, respectively $(P_2(t))$ is: the objects are the pairs $\{(i, s), i \in \{1, ..., I\}, s \in \{1, ..., K\}\}$, respectively $\{(i, s), i \in \{I+1, ..., L\}, s \in \{1, ..., K\}\}$; a class consists of the objects corresponding to the same segment; the profit of an object (i, s) is $u_i(R_s)$ and its size is $V(R_s)n_i(\alpha + l_i t)$ for $i \in \{1, ..., I\}$, respectively $V(R_s)n_i(\alpha t + l_i)$ for $i \in \{I+1, ..., L\}$. The volumes of the knapsacks are 1, respectively t. In $(P_1(t))$ an extra condition is imposed, namely that the zero rate cannot be allocated to all users in cell X.

Since $(P_1(t))$ and $(P_2(t))$ are multiple choice knapsack problems, close to optimal solutions can be found by applying for example the FPTAS described in [CHW76]. For an $\epsilon > 0$ and $t \in [t_{\min}, t_{\max}]$, let $K_1(t,\epsilon)$ and $K_2(t,\epsilon)$, be the value of the solution given by a FPTAS for $P_1(t)$, respectively $P_2(t)$. Hence,

$$K_1(t,\epsilon) \geq (1-\epsilon)OPT_1(t),$$

and

$$K_2(t,\epsilon) \geq (1-\epsilon)OPT_2(t).$$

Let t^* be the value for which $OPT_1(t^*) + OPT_2(t^*) = OPT$.

In the next lemma we will prove that a feasible solution of (P') of value at least $(1-\epsilon)OPT$ can be found using only the values $K_1(t,\epsilon)$ and $K_2(t,\epsilon)$, for $t \in [t_{min}, t_{max}]$.

Lemma 3.3.2 *For each $\epsilon > 0$, the following relation holds*

$$\max_{t \in [t_{min}, t_{max}]} \{K_1(t,\epsilon) + K_2(t,\epsilon)\} \geq (1-\epsilon)OPT.$$

Proof. From Lemma 3.3.1 follows

$$\begin{aligned}\max_{t \in [t_{min}, t_{max}]} \{K_1(t,\epsilon) + K_2(t,\epsilon)\} &\geq K_1(t^*,\epsilon) + K_2(t^*,\epsilon), \\ &\geq (1-\epsilon)OPT_1(t^*) + (1-\epsilon)OPT_2(t^*), \\ &\geq (1-\epsilon)OPT,\end{aligned}$$

where for the second inequality we have used that $K_1(t^*,\epsilon)$, respectively $K_2(t^*,\epsilon)$ are values returned by a FPTAS for $(P_1(t^*))$, respectively $(P_2(t^*))$. ■

However, if $\epsilon \geq \frac{1}{2}$, in order to find a solution of value $(1-\epsilon)OPT$ it is not necessary to calculate $\max_{t \in [t_{min}, t_{max}]} \{K_1(t,\epsilon) + K_2(t,\epsilon)\}$. Let $r = \{r_1, ..., r_I\}$ and $r' = \{r_{I+1}, ..., r_L\}$ be two rate allocations that give a total utility for cell 1, respectively cell 2, of value at least $\frac{1}{2}OPT_1(t_{min})$, respectively $\frac{1}{2}OPT_2(t_{max})$. Since $OPT_1(t)$ is a decreasing function and $OPT_2(t)$ is an increasing function, it follows that the rate allocation $r'' = (r_1, ..., r_I, r_{I+1}, ..., r_L)$ gives a total utility of value at least $\frac{1}{2}OPT$. The rate allocations r and r' with the above mentioned properties can be found by applying standard methods (see [CHW76]).

In the sequel, we suppose that $\epsilon < \frac{1}{2}$.

The only bottleneck in finding a solution of (P') of value at least $(1-\epsilon)OPT$ is that we have to calculate $K_1(t,\epsilon)$ and $K_2(t,\epsilon)$ for all $t \in [t_{min}, t_{max}]$. However, as we will see below, we can still obtain a solution close to optimum by analysing only a polynomial number of values of t.

For $\epsilon > 0$, let t_{app} be the value of t for which

$$K_1(t_{app},\epsilon) + K_2(t_{app},\epsilon) = \max_{t\in[t_{min},t_{max}]}\{K_1(t,\epsilon) + K_2(t,\epsilon)\}. \tag{3.14}$$

Note that $OPT_1(t)$, respectively $OPT_2(t)$ are step functions and have at most 2^{KI}, respectively 2^{KJ} jump points, the number of the possible rate assignments in each cell. Therefore, for finding t_{app}, it would suffice to check only the jump points of the two functions.

Next lemma further reduce the set of t's that must be considered for obtaining a solution of value at least $(1-\epsilon)OPT$.

Lemma 3.3.3 *For each* $\epsilon < \frac{1}{2}$, *the following holds*

$$t_{app} \in [t_{min}, t_{max}] \setminus \{t | K_1(t_{app},\epsilon) < \epsilon K_1(t_{min},\epsilon) \text{ and } K_2(t_{app},\epsilon) < \epsilon K_2(t_{max},\epsilon)\}.$$

Proof. Suppose that $K_1(t_{app},\epsilon) < \epsilon K_1(t_{min},\epsilon)$ and $K_2(t_{app},\epsilon) < \epsilon K_2(t_{max},\epsilon)$. Hence,

$$K_1(t_{app},\epsilon) + K_2(t_{app},\epsilon) < \epsilon\left(K_1(t_{min},\epsilon) + K_2(t_{max},\epsilon)\right),$$

which, since $\epsilon < \frac{1}{2}$, leads to a contradiction with

$$\begin{aligned} K_1(t_{app},\epsilon) + K_2(t_{app},\epsilon) \geq \quad & \tfrac{1}{2}\left(K_1(t_{min},\epsilon) + K_2(t_{min},\epsilon)\right. \\ & \left.+K_1(t_{max},\epsilon) + K_2(t_{max},\epsilon)\right). \end{aligned}$$

■

Consider the sets $A_l(\epsilon)$ and $\overline{A_l}(\epsilon)$, for $l \in \{0, 1, ..., \lfloor \frac{1}{\epsilon}\ln\frac{1}{\epsilon}\rfloor + 1\}$ defined as

$$\begin{aligned} & A_0(\epsilon) = \{t | K_1(t_{min},\epsilon) < K_1(t,\epsilon)\}, \\ & \overline{A_0}(\epsilon) = \{t | K_2(t_{max},\epsilon) < K_2(t,\epsilon)\}, \\ & A_l(\epsilon) = \{t | (1-\epsilon)^l K_1(t_{min},\epsilon) < K_1(t,\epsilon) < (1-\epsilon)^{l-1} K_1(t_{min},\epsilon)\}, \text{ for } l \geq 1, \\ & \overline{A_l}(\epsilon) = \{t | (1-\epsilon)^l K_2(t_{max},\epsilon) < K_2(t,\epsilon) < (1-\epsilon)^{l-1} K_2(t_{max},\epsilon)\}, \text{ for } l \geq 1. \end{aligned}$$

Remark 3.3.1 From the fact that $(1-\epsilon)^{\frac{1}{\epsilon}\ln\frac{1}{\epsilon}} < \epsilon$, and from Lemma 3.3.3 follows that $t_{app} \in \bigcup_{l=0}^{\lfloor \frac{1}{\epsilon}\ln\frac{1}{\epsilon}\rfloor+1} (A_l(\epsilon) \cup \overline{A_l}(\epsilon))$

Further we will prove that by choosing only one element from each set A_l, respectively $\overline{A_l}$, we will not deviate significantly from the optimum. This will reduce the number of t's to consider to at most $\lfloor \frac{2}{\epsilon}\ln\frac{1}{\epsilon}\rfloor + 2$.

Lemma 3.3.4 *Recall t_{app} defined in 3.14 and the sets $A_l(\epsilon)$ and $\overline{A_l}(\epsilon)$, then*
a) If $t_{app} \in A_l(\epsilon)$, then for each $t \in A_l(\epsilon)$,$(1-\epsilon)K_1(t_{app},\epsilon) \leq K_1(t,\epsilon)$.
b) If $t_{app} \in \overline{A_l}(\epsilon)$, then for each $t \in \overline{A_l}(\epsilon)$, $(1-\epsilon)K_2(t_{app},\epsilon) \leq K_2(t,\epsilon)$.

Proof. a) For $l = 0$,

$$K_1(t_{min},\epsilon) \geq (1-\epsilon)OPT_1(t_{min}) \geq (1-\epsilon)OPT_1(t_{app}) \geq (1-\epsilon)K_1(t_{app},\epsilon),$$

where for the second inequality we used the monotonicity of OPT_1. For $l \in \{1, ..., \lfloor \frac{1}{\epsilon}\ln\frac{1}{\epsilon} \rfloor + 1\}$ the proof follows immediately from the definition of A_l. ■

Let $J_1(\epsilon)$ be the set containing the maximal element from each nonempty set $A_l(\epsilon)$ and $J_2(\epsilon)$ the set containing the minimal element from each nonempty set $\overline{A_l}(\epsilon)$.

The following lemma shows that in order to find a feasible solution of (P) of value at least $(1-\epsilon)OPT$ it is enough to calculate $K_1(t,\epsilon')$ and $K_2(t,\epsilon')$ only for $t \in J_1(\epsilon') \cup J_2(\epsilon')$, for a well chosen ϵ'.

Lemma 3.3.5 *For $\epsilon' = 1 - \sqrt[3]{1-\epsilon}$ the following relation holds*

$$\max_{t \in J_1(\epsilon') \cup J_2(\epsilon')} \{K_1(t,\epsilon') + K_2(t,\epsilon')\} \geq (1-\epsilon)OPT.$$

Proof. We have seen in Remark 3.3.1 that

$$t_{app} \in \bigcup_{l=0}^{\lfloor \frac{1}{\epsilon'}\ln\frac{1}{\epsilon'} \rfloor + 1} (A_l(\epsilon') \cup \overline{A_l}(\epsilon')).$$

Suppose that $t_{app} \in A_k(\epsilon') \cap \overline{A_l}(\epsilon')$.
Let $t_k = J_1(\epsilon') \cap A_k(\epsilon')$ and $\overline{t_l} = J_2(\epsilon') \cap \overline{A_l}(\epsilon')$.

From Lemma 3.3.4 follows that

$$K_1(t_k,\epsilon') \geq (1-\epsilon')K_1(t_{app},\epsilon') \tag{3.15}$$

and

$$K_2(\overline{t_l},\epsilon') \geq (1-\epsilon')K_2(t_{app},\epsilon'). \tag{3.16}$$

Suppose that $t_k \geq \overline{t_l}$. Since $OPT_2(t)$ is an increasing function, the following relations hold:

$$\begin{aligned} K_2(t_k,\epsilon') &\geq (1-\epsilon')OPT_2(t_k) \geq (1-\epsilon')OPT_2(\overline{t_l}) \\ &\geq (1-\epsilon')K_2(\overline{t_l},\epsilon'). \end{aligned} \tag{3.17}$$

Combining (3.15), (3.16), (3.17) and Lemma 3.3.2, we obtain

$$\begin{aligned} K_1(t_k,\epsilon') + K_2(t_k,\epsilon') &\geq (1-\epsilon')(K_1(t_k,\epsilon') + K_2(\overline{t_l},\epsilon')), \\ &\geq (1-\epsilon'^2\left(K_1(t_{app},\epsilon') + K_2(t_{app},\epsilon')\right), \\ &\geq (1-\epsilon'^3 OPT), \end{aligned}$$

where the first inequality follows from (3.15), the second from (3.16) and (3.17), and the third from Lemma 3.3.2. Substituting $\epsilon' = 1 - \sqrt[3]{1-\epsilon}$ in the last relation, we get

$$\max_{t \in J_1(\epsilon') \cup J_2(\epsilon')} \{K_1(t,\epsilon') + K_2(t,\epsilon')\} \geq (1-\epsilon)OPT.$$

A similar analysis can be done if $t_k \leq \overline{t_l}$, but based on the monotonicity of $OPT_1(t)$.
■

Hence, the number of points we are looking at in order to find a solution close to the optimum is reduced to $|J_1(\epsilon)| + |J_2(\epsilon)| = \frac{2}{\epsilon'} \ln \frac{1}{\epsilon'} + 2 = O(\frac{1}{\epsilon'} \ln \frac{1}{\epsilon'}) = O(\frac{1}{\epsilon} \ln \frac{1}{\epsilon})$. Note that the points in $J_1(\epsilon') \cup J_2(\epsilon')$ can be found while running the FPTAS presented in [CHW76] for obtaining $K_1(t_{min}, \epsilon')$, respectively for $K_2(t_{max}, \epsilon')$. This implies that the following procedure is a FPTAS for problem (P):

Algorithm 1

1.	Let $\epsilon' = 1 - \sqrt[3]{1-\epsilon}$.
2.	Find the sets $J_1(\epsilon')$ and $J_2(\epsilon')$.
3.	For all $t \in J_1(\epsilon') \cup J_2(\epsilon')$, calculate $K_1(t,\epsilon')$ and $K_2(t,\epsilon')$, by using a FPTAS for the multiple choice knapsack problem.
4.	Choose the $t \in J_1(\epsilon') \cup J_2(\epsilon')$ for which $\max_{t \in J_1 \cup J_2} \{K_1(t,\epsilon') + K_2(t,\epsilon')\}$ is attained.
5.	Return the rate allocation obtained by solving $K_1(t_{app}, \epsilon')$ and $K_2(t_{app}, \epsilon')$.

If, for solving the multiple choice knapsack problems, one uses the FPTAS described in [CHW75], which, for a given ϵ, runs in time $O(\frac{K^3 L}{\epsilon})$, then the running time of the algorithm presented above is $O(\frac{K^3 L}{\epsilon^2} \ln \frac{1}{\epsilon})$.

We conclude this section with several remarks on the algorithm.

Remark 3.3.2 The rate allocation provided in this chapter should be seen as an almost optimal allocation (with respect to the utility functions) in an ideal setting. Most notably, it requires the base stations to have perfect and complete information on location and path loss of the mobile users. This information is clearly not available at the base station. Implementation of rate allocation in a UMTS system will most likely be based on heuristics that use an approximation of location and path loss. For example, from the required power the base station can approximate the location and path loss. In order to characterize the performance of such a heuristic and of a rate allocation, one can use as a benchmark the ideal solution proposed in this chapter.

Remark 3.3.3 The aim of this chapter is to demonstrate that the rate allocation problem reduces to solving coupled multiple choice knapsack problems. For solving such knapsack problems, various approaches are available in the literature. If one is not interested in obtaining a FPTAS for the rate allocation problem, one can use other approximations or exact algorithms described in the literature (see *e.g.* [DRW95] for a fast branch and bound algorithm). Clearly, any algorithm for the multiple choice knapsack problem, should take into account the specific choice for the utility function. An extensive treatment of the influence of the utility function on the efficiency of the algorithms for solving the multiple choice knapsack problem is beyond the scope of this thesis.

Remark 3.3.4 Note that the rate allocation algorithm proposed above can be easily adapted to the case where, for each segment, a different set of rates are required by users. The only change will be in the definition of the classes in the underlying multiple choice knapsack problems. More precisely, if, for a segment i, only the rates in the set $\{R_{k_1}, ..., R_{k_2}\}$, with $k_1, k_2 \in \{1, ..., K\}$ are required, the class of objects corresponding to segment i will become $\{(i, s), s \in \{k_1, ..., k_2\}\}$.

Remark 3.3.5 The algorithm presented considers differentiated rate allocation in a two cell UMTS system, which goes beyond results described in the literature that usually consider the single cell case (see [DNZ02] and [Lit03]). For a UMTS network that covers a road, which is the main application intended in this chapter, interference among cells will be most likely restricted to neighbouring cells. The main bottleneck in applying our results for general networks, taking into account interference among more than two cells, is the explicit formula for the Perron-Frobenius eigenvalue that is underlying our decomposition among cells. Developing heuristics for more general networks, based on our results, seems possible.

3.4 Numerical examples

In the numerical examples, we use the following system parameters of the Wideband-CDMA system [HT07]: the system chip rate $W = 3.84$ MHz, thermal noise $N_0 = -169$ dBm/Hz, path loss exponent $\gamma = 4$, downlink non-orthogonality $\alpha = 0.3$, QoS required $\frac{E_b}{I_0}$, $\epsilon^* = 5$ dB, uplink transmission rate $r_U = 14$ kbps, downlink transmission rate $r_i \in \{14, 32, 64, 144\}$ kbps.

We first consider two BTSs with a non-homogeneous traffic load as shown in Figure 3.1. The distance between the two BTSs X and Y is 2000 meter. We divide the cells into 40 segments of width 50 meters. There is a block-shaped traffic jam, called as a hot spot, located at 650m from BTS X. The hot spot, i.e., the block-shaped traffic jam, has 10 segments, where the load in each segment is at most $\rho_s = 10$ Erlang. The load of a segment outside the hot spot is at most $\rho_s = 1$ Erlang. The system is overloaded, i.e., not all calls can be assigned a positive rate. For this typical traffic load, we investigate the optimal border location and downlink rate allocation obtained from (2.47) using Algorithm 1.

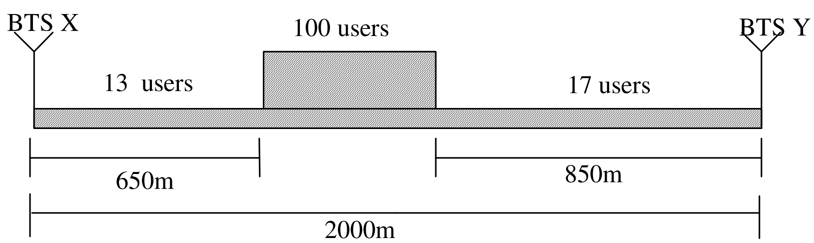


Figure 3.1: *Rectangular hot spot*

We solve the first stage problem using the algorithm for uplink optimal border location presented in [EvdBB05]. Figure 3.2 depicts the optimal border locations. From the figure we conclude that the optimal uplink cell borders are obtained if the uplink cell borders are located between the vertical lines in Figure 3.2. Thus, the optimal border for cell X is at 850m from BTS X and for cell Y is at 1000m from BTS Y.

Figure 3.3 depicts the optimal number of uplink users in cell X and cell Y. Notice that there is a coverage gap between BTS X and Y, which means that in order to maintain uplink feasibility some users have to be dropped.

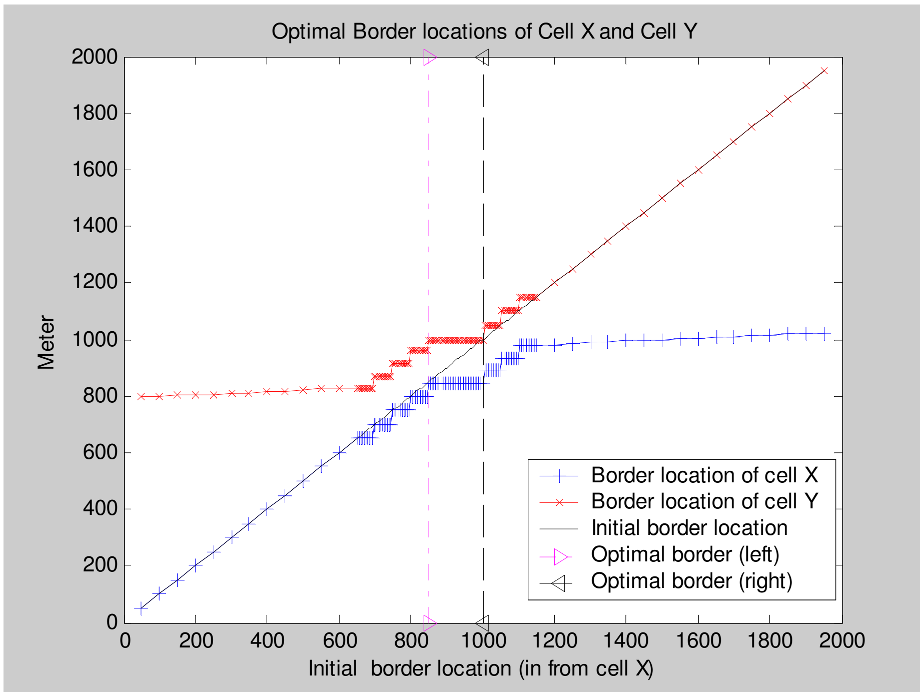


Figure 3.2: *Optimal border location*

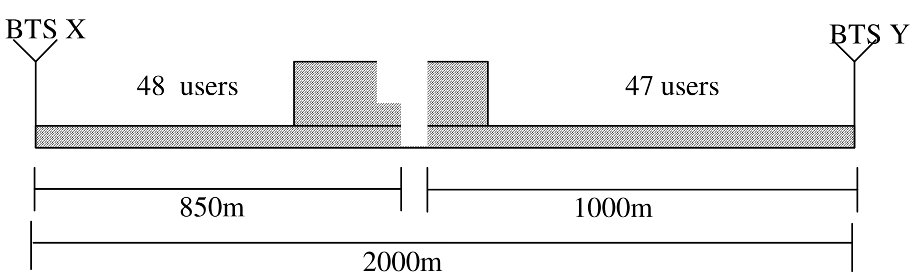


Figure 3.3: *Optimal number of uplink users*

Next, we investigate the FPTAS for the downlink rate allocation problem. Given the optimal uplink border from the first stage, we determine a downlink rate allocation which is close to the optimum. For finding a feasible solution of $(P1(t))$ and $(P2(t))$, we use Algorithm 1. For finding a feasible solution of the multiple-choice knapsack problems involved, we use a FPTAS based on dynamic programming as described in [CHW76, MT90]. We choose $\epsilon = 0.1$, i.e., we are interested in obtaining a solution of value at least $90\% * OPT$.

We consider two cases of the rate allocation according to the required transmission rates per segment. Firstly, we will consider the case where all segment in the cells can choose rate r_i from the same set of possible transmission rates $R_i \subseteq \{14 \text{ kbps}, 32 \text{ kbps}, 64 \text{ kbps}, 144 \text{ kbps}\}$. Secondly, we will consider the case where each segment only request some rates, i.e., each segment has different set of possible transmission rates $R_i \subset \{14 \text{ kbps}, 32 \text{ kbps}, 64 \text{ kbps}, 144 \text{ kbps}\}$.

Case I:

In case I, all segment in the cells can choose rate r_i from the same set of possible transmission rates $R_i \subseteq \{14 \text{ kbps}, 32 \text{ kbps}, 64 \text{ kbps}, 144 \text{ kbps}\}$.

First, we find the sets $J_1(\epsilon)$ and $J_2(\epsilon)$. We use the optimal uplink users as in Figure 3.3), i.e., the border of cell X is at 850m from BTS X and border of cell Y is at 1000m from BTS Y. Figure 3.4 and Figure 3.5 depict the numerical results of the sets $J_1(\epsilon)$ and $J_2(\epsilon)$.

The next step is to find the maximum value of the total system utility, i.e., $\max\limits_{t \in J_1(\epsilon') \cup J_2(\epsilon')} \{K_1(t, \epsilon') + K_2(t, \epsilon')\}$. The maximum utility is attained at $t = 0.0628$ with value of 4784 units. The related rate allocation is shown in Figure 3.6, i.e., for cell X: 1 user with rate 32 kbps, 2 users with rate 64 kbps and 29 users with rate 144 kbps, and 16 users are dropped (receive 0 rate); and for cell Y: 2 users with 144 kbps, 5 users with rate 32 kbps and 40 users are dropped (receive 0 rate).

It can be seen that the maximum utility is attained by allocating maximum rate to most of the users in cell X which are close to BTS X and only few users in cell Y have a non zero rate. This confirms the intuitive rate allocation in the interference limited system, i.e., as the main interference sources are users from the other cell, it is optimal to allocate rate only to one of the cells at a time. Note that this numerical example describes an extreme situation, when cell X is heavily loaded. Therefore, after allocating rates to cell X, few resources remain available for cell Y, resulting in a small number of users in cell Y with a non-zero rate. In the case of less loaded cells, the number of users with non-zero rate in cell Y increases.

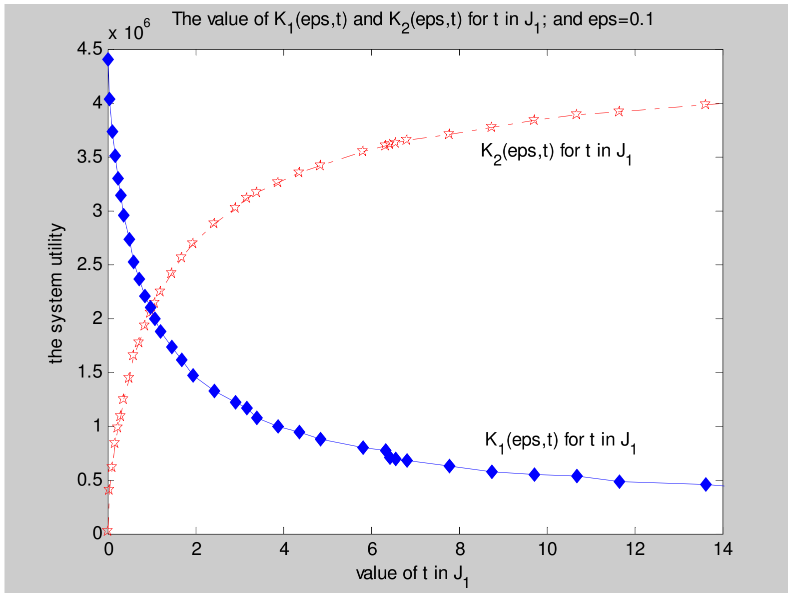


Figure 3.4: *$K_1(t, \epsilon)$ and $K_2(t, \epsilon)$ for $\epsilon = 0.1$ and $t \in J_1$*

.

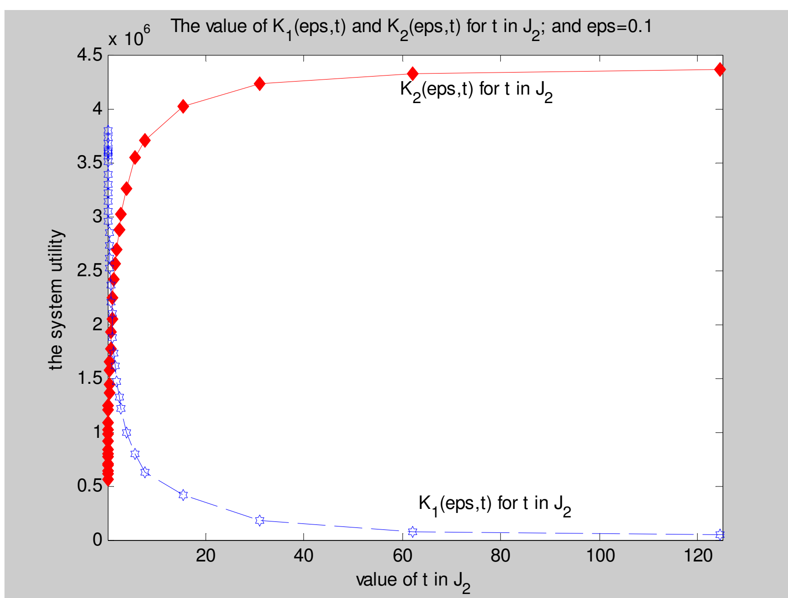


Figure 3.5: *$K_2(t, \epsilon)$ and $K_1(t, \epsilon)$ for $\epsilon = 0.1$ and $t \in J_2$*

Case II:

In the second case, we consider the case where each segment has different set of possible transmission rates $R_i \subset \{14 \text{ kbps}, 32 \text{ kbps}, 64 \text{ kbps}, 144 \text{ kbps}\}$.

Suppose users in cell X require a service with the following rates: users in segment $i = 1, \cdots, 13$ require service with rate either 14 kbps or 32 kbps; users in segment $i = 14, \cdots, 33$ require a service with rate either 32 kbps or 64 kbps; users in segment $i = 34, \cdots, 43$ require a service with 144 kbps and users in segment $i = 44, \cdots, 48$ require a service with either 64 kbps or 144 kbps. In cell Y all users require a service with rate either 32 kbps or 64 kbps. The maximum system utility is again approximated by $\max_{t \in J_1(\epsilon') \cup J_2(\epsilon')} \{K_1(t, \epsilon') + K_2(t, \epsilon')\}$.

The maximum utility is attained at $t = 0.0020$ with value of 7424 units. The related rate allocation is shown in Figure 3.7, i.e., for cell X: 13 users with rate 32 kbps, 49 users with rate 64 kbps and 26 users with rate 144 kbps; and for cell Y: 2 users with 64 kbps, 6 users with rate 32 kbps and 39 users are dropped (receive 0 rate).

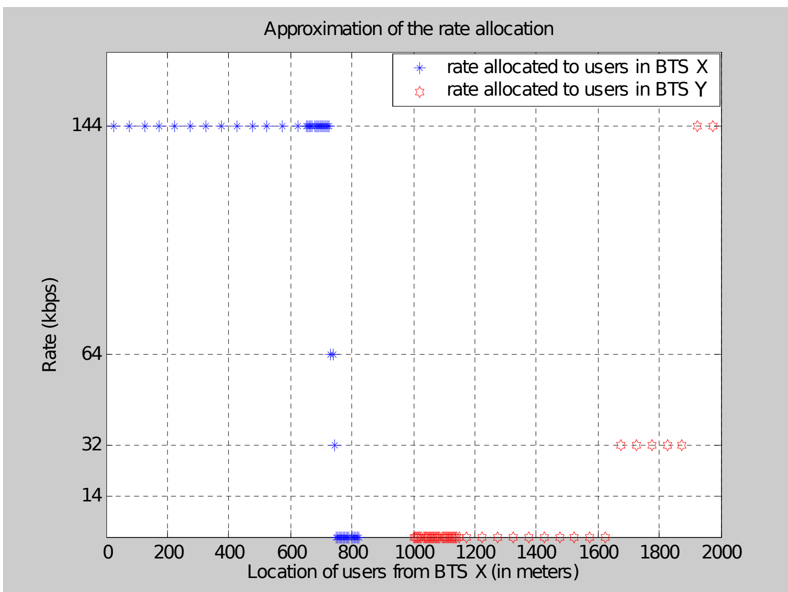


Figure 3.6: *Rate allocation for case 1*

Notice that by restricting the set of available rates in a segment to the set of requested rates, a higher system utility is obtained (4784 in case I versus 7424 in case II) and less users are dropped (56 users in case I versus 39 users in case II).

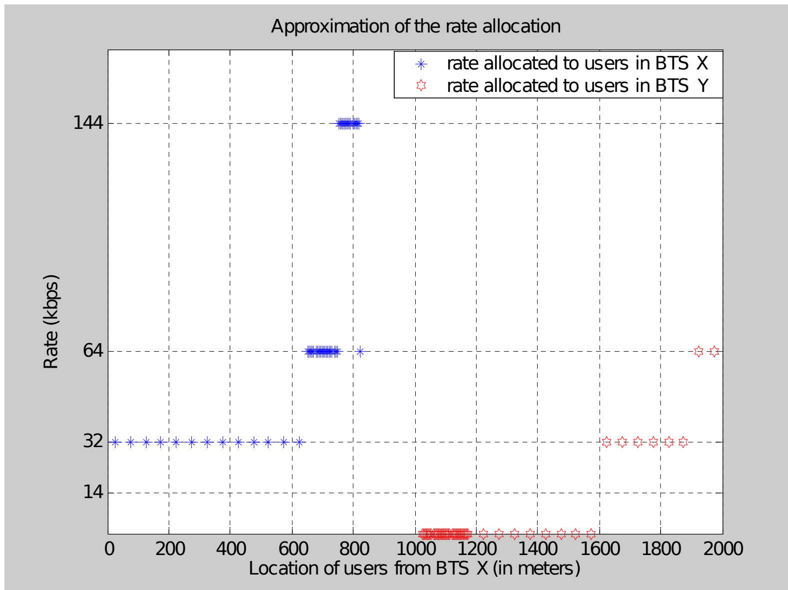


Figure 3.7: *Rate allocation for case 2*

3.5 Conclusions

This chapter has provided a combinatorial algorithm for finding a downlink rate allocation in a CDMA network, that, for an $\epsilon > 0$, achieves a throughput of value at least $(1 - \epsilon)$ times the optimum. Based on the Perron-Frobenius eigenvalue of the power assignment matrix, we have reduced the downlink rate allocation problem to a set of multiple-choice knapsack problems, for which efficient algorithms are known. This approach proves to have several advantages. First, the discrete optimization approach has eliminated the rounding errors due to continuity assumptions of the downlink rates. Using our model, the exact rate that should be allocated to each user can be indicated. Second, the rate allocation approximation we proposed guarantees that the solution obtained is close to the optimum. Moreover, the algorithm works for very general utility functions. Furthermore, our results indicate that the optimal downlink rate allocation can be obtained in a distributed way: the allocation in each cell can be optimized independently, interference being incorporated in a single parameter t.

Chapter 4

Two-Cell: Exact Algorithm for Optimal Joint Rate and Power Allocation

This chapter addresses, in an analytical setting, the joint power and rate assignment in two cell in a CDMA network. The assumption made in this chapter is that users' data rates can be assigned from the continuous interval $[r_{\min}, R_{\max}]$, with $r_{\min} > 0$.

4.1 Model

We consider a system with mobile users served by two base transmitter stations (BTS), X and Y, as in the previous chapter. Rather than discretizing the cells into small segments, we allow the users' location to take continuous values in the cell. Moreover, we allow the rates to take continuous values in the interval. Denote by U_X, respectively U_Y, the set of mobiles served by BTS X, respectively BTS Y. Let $l_{i,X}$ denote the path loss from BTS X to mobile i. We assume that mobiles are served by a single BTS.

Let ϵ_i denote the energy per bit to interference ratio requirement for mobile i. Let P_{iX} denote the transmission power of BTS X to mobile i. A configuration of mobiles is feasible when for each mobile i served by BTS X, say, the energy per bit to interference ratio exceeds the threshold ϵ_i. If a configuration is feasible, then under perfect power control the energy per bit to interference ratio $\left(\frac{E_b}{I_0}\right)_i$ equals this threshold. Thus, assuming perfect power control, feasibility for a configuration in which mobile i is served by BTS X is characterized by,

$$\left(\frac{E_b}{I_0}\right)_i := \frac{W}{r_i}\frac{P_{iX}l_{i,X}}{\alpha l_{i,X}(\sum_{j\in U_X} P_{jX} - P_{iX}) + l_{i,Y}\sum_{j\in U_Y} P_{jY} + N_i} = \epsilon_i, \tag{4.1}$$

where U_X is the set of mobiles served by BTS X, W is the system chip rate, α is the downlink orthogonality factor, and r_i is the data rate for mobile i, and N_i be the thermal noise at the location of mobile i,

Data rates can be assigned from the continuous interval $[r_{\min}, R_{\max}]$, with $r_{\min} > 0$. The optimization problem is to determine an assignment of rates and powers to mobiles that maximizes the total rate.

For each fixed number of mobile calls placed in the coverage area, the rate assignment problem can be formulated as the following optimization problem:

$$\begin{aligned}
(P) \qquad \max \quad & \sum_{i\in U_X\cup U_Y} r_i \\
\text{s.t.} \quad & \left(\frac{E_b}{I_0}\right)_i = \epsilon_i, \quad i\in U_X\cup U_Y, \\
& \sum_{i\in U_X} P_{iX} \le P_X^{\max}, \\
& \sum_{i\in U_Y} P_{iY} \le P_Y^{\max}, \\
& r_i \in [r_{\min}, R_{\max}], \quad i\in U_X\cup U_Y, \\
& P_{iX} \ge 0, \quad \forall i \in U_X\cup U_Y.
\end{aligned}$$

Next, we derive the characterisation of the optimal rate assignment.

4.2 Characterization of an optimal rate assignment

Let $x = \sum_{i \in U_X} P_{iX}$ and $y = \sum_{i \in U_Y} P_{iY}$. And let $V(r_i) = \epsilon r_i/(W + \alpha \epsilon r_i)$. From 4.1, the optimization problem (P) can be rewritten as:

$$
\begin{aligned}
(P) \quad \max \quad & \sum_{i \in U} r_i \\
\text{s.t.} \quad & \left(1 - \alpha \sum_{i \in U_X} V(r_i)\right) x - \sum_{i \in U_X} V(r_i) l_i y - \sum_{i \in U_X} V(r_i) l_{i,X}^{-1} N_i = 0, && (4.2) \\
& - \sum_{i \in U_Y} V(r_i) l_i x + \left(1 - \alpha \sum_{i \in U_Y} V(r_i)\right) y - \sum_{i \in U_Y} V(r_i) l_{i,Y}^{-1} N_i = 0, && (4.3) \\
& P_X^{\max} - x \geq 0, && (4.4) \\
& P_Y^{\max} - y \geq 0, && (4.5) \\
& x \geq 0, && (4.6) \\
& y \geq 0, && (4.7) \\
& R_{\max} - r_i \geq 0, \ \text{for } i \in U_X \cup U_Y, && (4.8) \\
& r_i - r_{\min} \geq 0, \ \text{for } i \in U_X \cup U_Y. && (4.9)
\end{aligned}
$$

Notice that this is neither a linear programming nor a convex programming problem. We assume that the rate assignment problem above has at least one feasible solution, or, in other words, that there exist powers x, y, such that assigning minimum rate to all users is feasible. Moreover, observe that in an optimal solution $(x^*, y^*, \mathbf{r}^*)$ of the optimization problem (P),

$$
\left(1 - \alpha \sum_{i \in U_X} V(r_i^*)\right) > 0 \qquad \text{and} \qquad \left(1 - \alpha \sum_{i \in U_Y} V(r_i^*)\right) > 0.
$$

For later reference, we also provide the Lagrangian. Let $\lambda \in \mathbb{R}^6$, μ, $\nu \in \mathbb{R}^{|U|}$ be the Lagrangian multipliers corresponding to equations (4.2)-(4.9).

Denote by $\mathbf{r} = (r_i)_{i \in U_X \cup U_Y}$ the vector of the rates allocated to users.

The Lagrangian corresponding to the optimization problem (P) is

$$\begin{aligned} L(x, y, \mathbf{r}, \lambda, \mu, \nu) = & \sum_{i \in U} r_i \\ & + \lambda_1((1 - \alpha \sum_{i \in U_X} V(r_i))x - \sum_{i \in U_X} V(r_i) l_i y - \sum_{i \in U_X} V(r_i) l_{i,X}^{-1} N_i)) \\ & + \lambda_2 \left(- \sum_{i \in U_Y} V(r_i) l_i x + (1 - \alpha \sum_{i \in U_Y} V(r_i))y - \sum_{i \in U_Y} V(r_i) l_{i,Y}^{-1} N_0 \right) \\ & + \lambda_3 (P_X^{max} - x) + \lambda_4 (P_Y^{max} - y) + \lambda_5 x + \lambda_6 y \\ & + \sum_{i \in U} \mu_i (R_{\max} - r_i) + \sum_{i \in U} \nu_i (r_i - r_{\min}). \end{aligned}$$

Next we give a monotonicity property of the rates.

Theorem 4.2.1 *If $(x^*, y^*, \mathbf{r}^*)$ is an optimal solution of the problem (P), then for any two calls i and j, say, in cell X,*

$$y^* l_i + l_{i,X}^{-1} N_i \le y^* l_j + l_{j,X}^{-1} N_j \Rightarrow r_i^* \ge r_j^*. \tag{4.10}$$

A similar statement holds for cell Y.

Proof. Suppose there exist two calls $i, j \in U_X$ such that

$$l_i y^* + l_{i,X}^{-1} N_i \le l_j y^* + l_{j,X}^{-1} N_j \tag{4.11}$$

and

$$r_i^* < r_j^*. \tag{4.12}$$

Define the following rate vector $\hat{\mathbf{r}} \in \mathbb{R}^{|U_X|+|U_Y|}$:

$$\hat{r_k} = \begin{cases} r_k^*, & \text{for } k \in U_X \cup U_Y \setminus \{i, j\} \\ r_j^*, & \text{for } k = i, \\ r_i^*, & \text{for } k = j, \end{cases}$$

i.e., with rate assignment to calls i and j interchanged. As the total rate is unchanged, the throughput of the rate assignments $\mathbf{r}$ and $\hat{\mathbf{r}}$ is the same. Let

$$\hat{x} = \frac{\sum_{i \in U_X} V(\hat{r_i})(l_i y^* + l_{i,X}^{-1} N_i)}{1 - \alpha \sum_{i \in U_X} V(\hat{r_i})}. \tag{4.13}$$

It can be easily seen that $\hat{x} < x^*$.

Note that $(\hat{x}, y^*, \hat{\mathbf{r}})$ is not necessarily a feasible solution of P, since it may not satisfy constraints (4.2) and (4.3). However, we can obtain a feasible solution by increasing the rates $\hat{r}$ for users in $U_X \setminus \{j\}$, until power x^* is reached in (4.13) or all rates in $U_X \setminus \{j\}$ are maximal.

Denote by $(\tilde{\mathbf{r}})_{U_X}$ the rate assignment obtained in this way. Suppose that

$$(\tilde{\mathbf{r}}_k)_{k \in U_X \setminus \{j\}} = (R_{\max})_{U_X \setminus \{j\}}. \tag{4.14}$$

By decreasing y^* to a value $\hat{y}$ such that $(\hat{x}, \hat{y}, (\tilde{\mathbf{r}}_k)_{k \in U_X \setminus \{j\}}, (\hat{\mathbf{r}}_k)_{k \in U_Y})$ such that (4.3) is satisfied, while the rates for users in U_Y remain the same, we obtain a feasible power/rate allocation with a higher throughput than $\mathbf{r}^*$.

If x^* was reached in (4.13), then

$$(x^*, y^*, (\hat{\mathbf{r}}_k)_{k \in U_X}, (\mathbf{r}^*_k)_{k \in U_Y}) \tag{4.15}$$

is a feasible solution of P which gives a higher throughput than

$$(x^*, y^*, (\mathbf{r}^*_k)_{k \in U_X}, (\mathbf{r}^*_k)_{k \in U_Y}). \tag{4.16}$$

This contradicts the fact that $(x^*, y^*, \mathbf{r}^*)$ is an optimal solution. Hence, it must be that $r_i^* \geq r_j^*$.
■

From theorem above, we have derived a monotonicity property of the optimal rate assignment. The next theorem described the optimal rate assignment in two cells.

Theorem 4.2.2 *If $(x^*, y^*, \mathbf{r}^*)$ is an optimal solution of the problem (P), then there exists an optimal power and rate assignment such that at most one mobile i in cell X has intermediate rate r_i^*, $r_{min} < r_i^* < R_{max}$. A similar statement holds for cell Y.*

Proof. Suppose that there exist two mobiles i and j with intermediate rate r_i^*, respectively r_j^* such that $r_{min} < r_j^* < R_{max}$ and $r_{min} < r_i^* < R_{max}$.

Let $a_i = y^* l_i + l_{i,X}^{-1} N_i$ and $a_j = y^* l_j + l_{j,X}^{-1} N_j$.

According to Theorem 4.2.2, if $a_i \leq a_j$, then we have $r_j^* \leq r_i^*$.

Now, suppose that $r_j^* < r_i^*$. Let $\alpha = \min\{R_{max} - r_i^*, r_j^* - r_{min}\}$.

Assume that $\alpha = R_{max} - r_i^*$. Since V is concave,

$$V(R_{max}) - V(r_i^*) = V(r_i^* + \alpha) - V(r_i^*) < V(r_j^*) - V(r_j^* - \alpha),$$

and consequently

$$V(R_{max})a_i + V(r_j^* - \alpha)a_j < V(r_j^*)a_j + V(r_i^*)a_i < V(R_{max})a_i + V(r_j^*)a_j.$$

Since V is continuous and increasing, there exist a β, $0 < \beta < \alpha$, such that

$$V(R_{max})a_i + V(r_j^* - \beta)a_j = V(r_j)a_j + V(r_i)a_i.$$

One can easily verify that the rate assignment given by $\hat{\mathbf{r}}$ defined as

$$\hat{r}_k = \begin{cases} r_k^*, \text{ for } k \in U_X \setminus \{i, j\}, \\ R_{max}, \text{ for } k = i, \\ r_j^* - \beta, \text{ for } k = j, \end{cases} \tag{4.17}$$

is feasible and it has a higher total rate than $\mathbf{r}^*$, since $\beta < \alpha$.

Similarly, if $\alpha = r_j^* - r_{min}$, one can show that there exists a β such that $r_j^* - r_{min} < \beta < R_{max} - r_i^*$ with the property that

$$\hat{r}_k = \begin{cases} r_k^*, \text{ for } k \in U_X \setminus \{i, j\}, \\ r_i^* + \beta, \text{ for } k = i, \\ r_{min}, \text{ for } k = j, \end{cases} \tag{4.18}$$

is feasible and it has a higher total rate than than $\mathbf{r}^*$. Hence, we can conclude that there exists at most one mobile with intermediate rate in one cell. ■

Next, we will derive the optimal rate assignment. First, we will simplify some notations of the optimization (P). Denote the functions in the left hand side of constraints (4.2)-(4.7) of the optimization problem (P), by

$$h_1(x, y, r), ..., h_6(x, y, r), \tag{4.19}$$

and denote the functions in the left hand side of constraints (4.8)-(4.9) of the optimization problem (P) by

$$g_1(x, y, r), ..., g_{2|U_X|+2|U_Y|}(x, y, r). \tag{4.20}$$

Next, we will review some optimization terminology (see [BNO03]).

- Active Constraint

  If an inequality constraint of (P) is satisfied with equality in a feasible vector $(x, y, r) \in \mathbb{R}^{|U_X|+|U_Y|+2}$ of (P), the constraint is *active* in (x, y, r). Denote by $A(x, y, r)$ the set of *active* inequalities in the point (x, y, r).

- Regular Point:

  A feasible vector (x, y, r) is *regular* if the gradients $\nabla h_1(x, y, r)$, $\nabla h_2(x, y, r)$ and $\nabla h_i(x, y, r)$, $\nabla g_j(x, y, r)$ for

  $$i \in A(x, y, r) \bigcap \{3, 4, 5, 6\}, \tag{4.21}$$

  and $j \in A(x, y, r)$ are linearly independent.

  Notice that $\nabla h_1(x, y, r)$, $\nabla h_2(x, y, r)$ are linearly independent for any feasible (x, y, r) , so that all points with $A(x, y, r) = \emptyset$ are regular.

- Further, note that since $r_{\min} > 0$, $x \neq 0$ and $y \neq 0$ in the optimal solution. Moreover, since the objective function is linear, each optimum must be a global optimum.

Next, we will start by characterizing the rate assignment for regular points. In the proofs that follow, we will make use of the Karush-Kuhn-Tucker (KKT) necessary conditions for a regular point to be an optimal solution (see [BNO03]). They state that for a regular point (x^*, y^*, r^*) that is an optimum of (P) there exists an unique multiplier vector $(\lambda^*, \mu^*, \nu^*)$ such that:

(K1) $\bigtriangledown_{(x^*,y^*,r^*)} L(x^*, y^*, r_i^*, \lambda^*, \mu^*, \nu^*) = 0$,where L denotes the Lagrangian function corresponding to the optimization problem (P) ,

(K2) $\lambda_k^* \geq 0$, for $k \in \{3, 4, 5, 6\}, \mu^* \geq 0$ and $\nu^* \geq 0$,

(K3) The Lagrangian multipliers corresponding to non active constraints are equal to 0.

Theorem 4.2.3 *If $(x^*, y^*, \mathbf{r}^*)$ is a regular optimal solution of the problem (P) then $x^* = P_X^{max}$ or $y^* = P_Y^{max}$ or $r_i^* = R_{\max}$, for each call $i \in U_X \cup U_Y$.*

Proof. Note that since the minimum rate can be ensured to all accepted users, constraints (4.2) and (4.3) imply that $x^* > 0$ and $y^* > 0$. Thus, based on condition (K3), we conclude that $\lambda_5^* = \lambda_6^* = 0$. Suppose that $x^* < P_X^{max}$, $y < P_Y^{max}$ and $r_{\min} \leq r_i < R_{\max}$ for a call $i \in U_X$.

From (K3), follows that $\lambda_3^* = \lambda_4^* = 0$ and that $\mu_i^* = 0$.

Moreover, (K1) implies that

$$\frac{\partial L}{\partial x}(x^*, y^*, r^*, \lambda^*, \mu^*, \nu^*) = 0,$$

$$\frac{\partial L}{\partial y}(x^*, y^*, r_i^*, \lambda^*, \mu^*, \nu^*) = 0,$$

and

$$\frac{\partial L}{\partial r_i}(x^*, y^*, r^*, \lambda^*, \mu^*, \nu^*) = 0. \tag{4.22}$$

Hence,

$$\begin{cases} \lambda_1^* \left(1 - \alpha \sum_{i \in U_X} V(r_i^*)\right) - \lambda_2^* \sum_{i \in U_Y} V(r_i^*) l_i & = 0 \\ -\lambda_1^* \sum_{i \in U_X} V(r_i^*) l_i + \lambda_2^* \left(1 - \alpha \sum_{i \in U_Y} V(r_i^*)\right) & = 0. \\ 1 + \nu_i^* - \mu_i^* - \lambda_1^* V'(r_i^*)(\alpha x^* + l_i y^* + l_{i,X}^{-1} N_i) & = 0. \end{cases} \tag{4.23}$$

Observe that the first two equations in λ_1^*, λ_2^* are linearly independent (recall constraints (4.2)-(4.3) and the assumption that a minimal rate assignment is feasible), so the only solution is $\lambda_1^* = \lambda_2^* = 0$.

Further, since $\mu_i^* = 0$, from the third equation in (5.11) follows that $\nu_i = -1$, which contradicts condition (K2), that $\nu_i^* \geq 0$.

Hence, in an optimal solution, either the rates of all users are maximal, or the power in one of the cells is maximal. ■

Corollary 4.2.4 *Let $(x^*, y^*, \mathbf{r}^*)$ be regular and an optimal solution of problem (P). Suppose that calls in cell X, respectively in cell Y are ordered in increasing order of their $l_i y^* + l_{i,X}^{-1} N_i$, respectively $l_j x^* + l_{j,Y}^{-1} N_j$ values. Then:*

- *there exists a positive number $A(y^*)$, such that*
  for each $i \in U_X$ with $l_i y^* + l_{i,X}^{-1} N_i < A(y^*)$, $r_i^* = R_{\max}$ and
  for each $i \in U_X$ with $l_i y^* + l_{i,X}^{-1} N_i > A(y^*)$, $r_i^* = r_{\min}$,
- *there exists a positive number $B(x^*)$, such that*
  for each $j \in U_Y$ with $l_j x^* + l_{j,Y}^{-1} N_j < B(x^*)$, $r_j^* = R_{\max}$ and
  for each $j \in U_Y$ with $l_j x^* + l_{j,Y}^{-1} N_j > B(x^*)$, $r_j^* = r_{\min}$.

For a non regular point, the following theorem gives a complete characterization of the optimal power and rate assignment.

Theorem 4.2.5 *For each non regular point $(x, y, \mathbf{r})$, the following conditions are satisfied:*
a) $x = P_X^{max}$ or $y = P_Y^{max}$,
b) If $x = P_X^{max}$ and $y \neq P_Y^{max}$, then $r_i \in \{r_{\min}, R_{\max}\}$, for each $i \in U_X$,
c) If $y = P_Y^{max}$ and $x \neq P_X^{max}$, then $r_i \in \{r_{\min}, R_{\max}\}$, for each $i \in U_Y$.

Proof. Let $(x, y, \mathbf{r})$ be a non regular point, feasible for the optimization problem (P). Consider the matrix M formed by the $\nabla h_1(x, y, r)$, $\nabla h_2(x, y, r)$ and $\nabla h_i(x, y, r)$, $\nabla g_j(x, y, r)$ for

$$i \in A(x, y, r) \bigcap \{3, 4, 5, 6\}, j \in A(x, y, r). \tag{4.24}$$

Let K be the number of active inequality constraints. Notice that for a non-regular point it must be that $K > 0$, since $\nabla h_1(x, y, r)$, $\nabla h_2(x, y, r)$ are linearly independent. Clearly, $2 \leq rank(M) \leq K + 2$.

a) Suppose that $x \neq P_X^{max}$ and that $y \neq P_Y^{max}$. In other words, the active inequality constraints correspond to the constraints on rates. Then, matrix M has the following form:

$$M = \begin{pmatrix} 1-\alpha\sum_{i\in U_X} V(r_i) & -\sum_{i\in U_X} V(r_i)l_i & A & \mathbf{0} \\ -\sum_{i\in U_Y} V(r_i)l_i & 1-\alpha\sum_{i\in U_Y} V(r_i) & \mathbf{0} & B \\ 0 & 0 & C & \mathbf{0} \\ 0 & 0 & \mathbf{0} & D \\ \end{pmatrix},$$

where the vectors $A \in \mathbb{R}^{|U_X|}$, $B \in \mathbb{R}^{|U_Y|}$ are defined as follows:

$$A = [-V'(r_i)(\alpha x + l_i y + l_{i,X}^{-1} N_i)]_{i\in U_X}, \qquad B = [-V'(r_i)(\alpha y + l_i x + l_{i,Y}^{-1} N_i)]_{i\in U_Y},$$

and the matrices

$$C \in \mathbb{R}^{|\{i\in U_X : g_i \in A(x,y,r)\}|} \times \mathbb{R}^{|\{i\in U_X\}|}, \tag{4.25}$$

$$D \in \mathbb{R}^{|\{i\in U_Y : g_i \in A(x,y,r)\}|} \times \mathbb{R}^{|\{i\in U_Y\}|}, \tag{4.26}$$

are obtained from the diagonal square matrices with diagonal

$$diag(\overline{C}) = [I_{\{r_i = r_{\min}\}}) - I_{\{r_i = R_{\max}\}}]_{\{i\in U_X\}},$$
$$diag(\overline{D}) = [I_{\{r_i = r_{\min}\}}) - I_{\{r_i = R_{\max}\}}]_{\{i\in U_Y\}},$$

by deleting all rows for which the diagonal elements equals zero, where $I_{\{a\}} = 1$ if a is true, and 0 otherwise.

Clearly, $rank(C) + rank(D) = K$. Since constraints $\nabla h_1(x, y, r)$, $\nabla h_2(x, y, r)$ are linearly independent, it follows that $rank(M) = K + 2$, which contradicts the fact that $(x, y, \mathbf{r})$ is non regular. Hence, in a non regular point, the power assigned to one of the cells has to be maximal.

b) Suppose that $x = P_X^{max}$ and $y \neq P_Y^{max}$ and that there exist $i \in U_X$ such that $r_{\min} < r_i < R_{\max}$. It can be proved that the rank of the matrix M is again $rank(M) = K + 2$, which contradicts the fact that $(x, y, \mathbf{r})$ is non regular.

c) The proof is similar to b). ■

4.3 An Exact Algorithm for the optimization problem (P)

Based on Theorems 4.2.1-4.2.5 and Corollary 4.2.4, we now propose on algorithm for finding the optimal solution of (P). The algorithm relies on a reduction of the optimization problem (P) to a series of optimization problems in $\mathbb{R}$. Notice that the algorithm is exact since it considers all points, both the regular and non regular points.

The exact algorithm for two cells consists of three major step. In Step 1, the algorithm assigns maximum rates to all users. If maximum rate to all users is feasible, then the optimal solution has been found. To check whether the maximum rate is feasible, one only has to check if the corresponding powers calculated from (4.2)-(4.3) satisfy $0 \le x \le P_X^{max}$ and $0 \le y \le P_Y^{max}$. If this is not the case, then in Step 2, the algorithm calculates the rate allocation achieving maximum throughput for the case when the power in cell X is maximal, respectively the power in cell Y is maximal. In Step 3, the algorithm will choose among these 2 allocations the one with higher throughput. Note that if the rates are known, from (4.1), (4.2) and (4.3), the powers of each user can be derived. The algorithm is summarized below.

Step 1: Assign maximum rate R_{max} to all users $i \in U_X$ and $j \in U_Y$.
if there exists a feasible power allocation, **then**
  return as optimal solution $r_i = R_{max}$ for all users $i \in U_X$ and $j \in U_Y$.
else
  Step 2: Assign maximum power in cell X, then calculate a rate allocation that gives maximum throughput in cells X and Y. Next, assign maximum power in cell Y, then calculate a rate allocation that gives maximum throughput in cells Y and X.
  Step 3: Choose among the feasible rate and power allocations obtained in Step 2 the one that gives maximum throughput.
end if

Notice that Step 1 and Step 3 are fairly obvious. We will describe Step 2 in greater detail.

First, we consider the case when in cell X the base station transmits at maximum power, *i.e.*, $x^* = P_X^{max}$. Based on Corollary 4.2.4, we will find $B(P_X^{max})$. In Step 2, the algorithm provides a reduction of the optimization problem (P) that is based on a search procedure to find the values $B(x^*)$ and $A(y^*)$ introduced in Corollary 4.2.4 to obtain the set of mobiles at which the rate drops from $R_{\max}$ to $r_{\min}$ in both cells. As the set of users for maximum power at cell X also depends on the power assignment in cell Y, these sets cannot be determined independently. Thus we propose the following search algorithm from Theorem 4.2.1 and Theorem 4.2.2.

- According to Theorem 4.2.2, there is at most one user j^* in cell Y with intermediate rate $r_{j^*} \in (r_{\min}, R_{\max})$. Let $B(P_X^{max}) = l_{j*}P_X^{max} + l_{j*,Y}^{-1}N_{j*}$ be the value that characterizes user j^*.
- From Theorem 4.2.1 and Theorem 4.2.2 follows that for each $j \in U_Y$ with $l_jP_X + l_{j,Y}^{-1}N_j < B(P_X^{max})$, $r_j^* = R_{\max}$ and for each $j \in U_Y$ with $l_jP_X + l_{j,Y}^{-1}N_j > B(P_X^{max})$, $r_j^* = r_{\min}$.
- Suppose U_Y^{max} is the set of users with rate $R_{\max}$, and let $s = |U_Y^{max}|$ be the number of users with rate $R_{\max}$. Suppose U_Y^{min} is the set of users with rate

$r_{\min}$, and let $v = |U_Y^{min}|$ be the number of users with rate $r_{\min}$. We know that there is at most one user with intermediate rate r_Y. The rate r_Y is unknown at this stage of the algorithm. The power assigned to cell Y, as a function of r_Y, can be determined from constraint (4.3), and is given by

$$y^*(r_Y) = \frac{Q_Y}{1 - \alpha \left(\sum_{j \in U_Y^{\max}} V(R_{\max}) + \sum_{j \in U_Y^{\min}} V(r_{\min}) + V(r_Y) \right)}, \tag{4.27}$$

where

$$Q_Y = V(R_{\max}) \left(\sum_{j \in U_Y^{\max}} l_j x + l_{j,Y}^{-1} N_j \right) + V(r_{\min}) \left(\sum_{j \in U_Y^{\min}} l_j x + l_{j,Y}^{-1} N_j \right)$$
$$+ V(r_Y) \left(l_{j^*} x + l_{j^*,Y}^{-1} N_j \right)$$

- Similarly, for a specific $y^*(r_Y)$, Theorem 4.2.2 implies that there is at most one user i^* in cell X with $r_i \in (r_{\min}, R_{\max})$ which is characterized by the value of $l_i y^*(r_Y) + l_{i,Y}^{-1} N_0^i$, say $A(y^*(r_Y))$.

  Then all $i \in U_X$ with $l_i y^*(r_Y) + l_{i,X}^{-1} N_i < A(y^*(r_Y))$, have rate $R_{\max}$ and all $i \in U_X$ with $l_i y^*(r_Y) + l_{i,X}^{-1} N_i > A(y^*(r_Y))$ have rate $r_{\min}$.

- Suppose U_X^{max} is the set of users with rate $R_{\max}$, and let $u = |U_X^{max}|$ is the number of users with rate $R_{\max}$. Suppose U_X^{min} is the set of users with rate $r_{\min}$, and let the intermediate rate is $r_X \in (r_{\min}, R_{\max})$. Then the rate r_X can be expressed from (4.2) as follows:

$$r_X\,(r_Y) = \left(\frac{W}{\epsilon} \right) \frac{(P_X^{\max} - H_{max} - H_{min})}{\left(\alpha H_{max} + \alpha H_{min} + \ l_{i^*} y + l_{i^*,X}^{-1} N_i \right)}. \tag{4.28}$$

where

$$H_{max} = V(R_{\max}) \sum_{i \in U_X^{\max}} \left(\alpha x + l_i y^*\,(r_Y) + l_{i,X}^{-1} N_i \right),$$
$$H_{min} = V(r_{\min}) \sum_{i \in U_X^{\min}} \left(\alpha x + l_i y^*\,(r_Y) + l_{i,X}^{-1} N_i \right).$$

Note that if $B(P_X^{max})$, $|U_Y^{max}|$ and $|U_Y^{min}|$ were known, then $|U_X^{max}|$ and $|U_X^{min}|$ were also known. Then, r_Y would be the only unknown. This suggests that by enumerating all the possible values of $B(P_X^{max})$, $|U_Y^{max}|$ and $||U_Y^{min}||$, the problem could be reduced to an optimization problem in one variable, r_Y. The optimization problem is not easy to formulate due to the fact that the value of r_Y, more precisely $y^*(r_Y)$, is a decision variable in the assignment of $R_{\max}$ and $r_{\min}$ to users in U_X

(see Corollary 4.2.5). However, it can be easily seen that only some values of $y^*(r_Y)$ induce a different rate allocation in cell X. Let

$$L = \{\frac{l_{j_1,X}^{-1} N_{j_1} - l_{j_2,X}^{-1} N_{j_2}}{l_{j_2} - l_{j_1}}, j_1, j_2 \in U_X\} \bigcap R^+.$$

Suppose that $L \neq \emptyset$. For all $y^*(r_Y) \in [L_i, L_{i+1})$ the ordering of mobiles in cell X, as determined by their value of $l_i y^*(r_Y) + l_{i,X}^{-1} N_i$ is the same, but for different intervals $[L_j, L_{j+1})$ this ordering may be different. Note that $V(r)$ is strictly increasing, so that $y^*(r_Y)$ is strictly increasing. As a consequence, each unknown $y^*(r_Y) \in [L_i, L_{i+1})$ yields a unique r_Y.
Hence, for $y^*(r_Y) \in [L_i, L_{i+1})$, P can be reduced to the following optimization problem in $\mathbb{R}$:

$$\begin{aligned} \max \quad & r_X(r_Y) + r_Y \\ s.t. \quad & y^*(r_Y) \leq P_Y^{\max}, \\ & y^*(r_Y) \in [L_i, L_{i+1}], \\ & r_X(r_Y) \in [r_{\min}, R_{\max}], \\ & r_Y \in [r_{\min}, R_{\max}]. \end{aligned} \tag{4.29}$$

Thus, the original rate optimization problem can be reduced to $O(|U_X|^2)$ optimization problems in $\mathbb{R}$, one for each interval $[L_i, L_{i+1})$.

If $L = \emptyset$, then the order of the users in U_X does not depend on $y^*(r_Y)$ and we obtain a similar optimization problem to (4.29), without the second constraint.

Note that the optimization problems (4.29) are constraint optimization problems in one variable, which can be easily solved.

4.4 Conclusions

In this chapter we have proposed an exact algorithm for the joint rate and power allocation problem in two cells of a CDMA network. We have analyzed several properties of the optimal solutions, based on which we have proposed a polynomial time algorithm for solving the problem. Our results can be extended to non-decreasing utility functions at the cost of a rather involved notation. The multicells algorithm will be presented in the next chapter.

Chapter 5

Multi-cell: Exact and Heuristic Algorithm for Throughput Maximization

This chapter presents a full analytical characterization of the optimal joint downlink rate and power assignment for maximal total system throughput in a multi cell CDMA network. The rest of the chapter is organized as follows. Sections 5.1 and 5.2 present analytical results for the optimal rate and power assignment. Sections 5.3 and 5.4 present an exact algorithm and a heuristic for rate and power assignment. Numerical results and examples are provided in Section 5.5.

5.1 The model: multi-cell with continuous rates

First, we introduce the mathematical model for a multi-cell CDMA network. We consider the downlink in a system with mobiles U served by a set $\mathcal{B}$ of N base transmitter stations (BTS). Let $l_{i,X}$ denote the path loss from BTS X to mobile i, N_0^i the thermal noise received by mobile i, and let ϵ_i denote the energy per bit to interference ratio requirement for mobile i. Let P_{iX} denote the transmission power of BTS X to mobile i, and P_X^{max} the maximum down link transmission power of BTS X. The power received by mobile i from BTS X is $P_{iX}^{rec} = P_{iX} l_{i,X}$.

We impose the natural assumption for rate and power assignment to moving mobiles: each mobile is served by a single BTS. Let U_X denote the set of mobiles served by BTS X and r_i the rate at which mobile i is served. We assume a minimum and maximum rate r_{min} and R_{max}, where $r_{min} > 0$, i.e., a rate assignment is feasible when all the mobiles are served with at least at a minimal rate.

A power and rate assignment $(\mathbf{P}, \mathbf{r})$, with

$$\mathbf{P} = (P_{i,X})_{i \in U_X,\ X \in \mathcal{B}} \text{ and } \mathbf{r} = (r_i)_{i \in U},$$

is *feasible* if

$$\sum_{i \in U_X} P_{iX} \leq P_X^{max}$$

for all $X \in \mathcal{B}$ and

$$r_i \in [r_{min}, R_{max}],$$

for all $i \in U$ such that the energy per bit to interference ratio, $(E_b/I_0)_i$, exceeds the threshold ϵ_i required for correct decoding of transmissions.

We assume perfect power control so that the energy per bit to interference ratio equals its threshold

$$\left(\frac{E_b}{I_0}\right)_i := \frac{W}{r_i} \frac{P_{iX} l_{i,X}}{\alpha l_{i,X}(P_X - P_{iX}) + \sum\limits_{Y \in \mathcal{B} \setminus \{X\}} l_{i,Y} P_Y + N_0^i} = \epsilon_i,$$

for all $i \in U_X$, where X is the BTS serving mobile i, W is the system chip rate, and α is the down link orthogonality factor.

In a UMTS system rates are selected from a discrete set. However, these rates may be rapidly modified in accordance with channel conditions, resulting in average rates that lie in an interval $[r_{min}, R_{max}]$. The optimization problem considered in this chapter is to determine a feasible assignment of powers (to BTSs) and rates (to mobiles) that maximizes the total throughput $\sum_{i \in U} r_i$:

$$\begin{aligned} \max \sum_{i \in U} r_i & \\ \text{s.t.} \left(\frac{E_b}{I_0}\right)_i = \epsilon_i, & \quad \text{for all } i \in U, \\ \sum_{i \in U_X} P_{iX} \leq P_X^{max}, & \quad \text{for all } X \in \mathcal{B}, \\ r_i \in [r_{min}, R_{max}], & \quad \text{for all } i \in U, \\ P_{iX} \geq 0, & \quad \text{for all } i \in U_X, \quad X \in \mathcal{B}. \end{aligned} \tag{5.1}$$

In this chapter, for clarity of presentation, we assume that all mobiles have the same energy per bit to interference ratio threshold $\epsilon_i = \epsilon(r) = \epsilon$. For generalisation, see Remark 5.2.1.

The next section provides a more compact characterization of feasible power and rate assignments. This is done by reformulating the conditions of (5.1).

5.2 Feasible rate and power assignment

Consider a fixed rate assignment $\mathbf{r} \in \mathbb{R}^{|U|}$, with $r_{min} \leq r_i \leq R_{max}$, for all $i \in U$. A power assignment $\mathbf{P} = (P_X)_{X \in \mathcal{B}}$ is *feasible for rate assignment* $\mathbf{r}$, if $(\mathbf{P}, \mathbf{r})$ is a feasible rate and power assignment, i.e., for all $i \in U_X$ and all $X \in \mathcal{B}$ the conditions of (5.1) are satisfied:

$$\begin{cases} P_{iX} = \alpha V(r_i) P_X + V(r_i) \sum_{Y \in \mathcal{B} \setminus \{X\}} l_{i,X}^Y P_Y + V(r_i) l_{i,X}^{-1} N_0^i, \\ \sum_{i \in U_X} P_{iX} \leq P_X^{max}, \\ P_{iX} \geq 0, \end{cases} \tag{5.2}$$

where

$$V(r) = \frac{\epsilon r}{W + \alpha \epsilon r},$$

and

$$l_{i,X}^Y = \frac{l_{i,Y}}{l_{i,X}},$$

for $i \in U_X, Y \in \mathcal{B} \setminus \{X\}$.

Remark 5.2.1 (Generalising ϵ to $\epsilon(r)$) *The energy per bit to interference ratio is typically decreasing in the rate r_i to achieve the same degree of reliability, see e.g. [HT07, chapter 8]. This may be included in our results without changing the formulation of the problem. To this end, notice that the restrictions in our optimization problem include the energy per bit to interference ratio only via the function $V(r) = \epsilon r/(W + \alpha\epsilon r)$, see (5.2). Generalising ϵ to $\epsilon(r)$ will not affect the structure of the optimization problem. The analysis below merely requires that*

$$V(r) = \epsilon(r) r/(W + \alpha \epsilon(r) r)$$

is continuous, increasing and concave.

Let $\mathcal{F}_1(\mathbf{r})$ denote the set of feasible power assignments with rate assignment $\mathbf{r}$:

$$\mathcal{F}_1(\mathbf{r}) = \{\mathbf{P} \in \mathbb{R}^{|U|} : \mathbf{P} \text{ satisfies the system of equations (5.2)}\},$$

and $\mathcal{R}_1$ the set of rates within the allowed range for which a feasible power assignment exists:

$$\mathcal{R}_1 = \{\mathbf{r} \in \mathbb{R}^{|U|} : \mathcal{F}_1(\mathbf{r}) \neq \emptyset \text{ and } r_{min} \leq r_i \leq R_{max} \text{ for all mobiles } i \in U\}.$$

The rate and power assignment problem can be now written as:

$$\max\{\sum_{i \in U} r_i : \mathbf{r} \in \mathcal{R}_1\}. \tag{P}$$

It is convenient to rephrase $\mathcal{R}_1$ in terms of total powers allocated to cells. To this end, consider the following system of equations obtained by summing the first equation of (5.2) over $i \in U_X$:

$$(1 - \alpha \sum_{i \in U_X} V(r_i))P_X - \sum_{i \in U_X} V(r_i) \sum_{Y \in \mathcal{B} \setminus \{X\}} l_{i,X}^{Y} P_Y - \sum_{i \in U_X} V(r_i) l_{i,X}^{-1} N_0^i = 0, \tag{5.3}$$

$$0 \leq P_X \leq P_X^{max}, \qquad \text{for all } X \in \mathcal{B}. \tag{5.4}$$

For fixed $\mathbf{r}$, let

$$\mathcal{F}_2(\mathbf{r}) = \{\mathbf{P} \in \mathbb{R}^{|U|} : \mathbf{P} \text{ satisfies (5.3) and (5.4)}\},$$

and let $\mathcal{R}_2$ the set of rates:

$$\mathcal{R}_2 = \{\mathbf{r} \in R^{|U|} : \mathcal{F}_2(\mathbf{r}) \neq \emptyset \text{ and } r_{min} \leq r_i \leq R_{max} \text{ for all mobiles } i \in U\}.$$

For two cells, it was shown in previous chapter and in [BBEW06] that $\mathcal{R}_1 = \mathcal{R}_2$. The following lemma extends this result to an arbitrary number of cells.

Lemma 5.2.1 $\mathcal{R}_1 = \mathcal{R}_2$.

Proof. First, consider a rate assignment $\mathbf{r} \in \mathbb{R}^{|U|}$ for which $\mathcal{F}_1 \neq \emptyset$. Let $\mathbf{P} \in \mathbb{R}^{|U|}$ be the positive solution of (5.2).

By adding the powers of all mobiles in each cell $X \in \mathcal{B}$, we obtain

$$P_X = \sum_{i \in U_X} \alpha V(r_i) P_X + \sum_{i \in U_X} V(r_i) \sum_{Y \in \mathcal{B} \setminus \{X\}} l_{i,X}^{Y} P_Y + \sum_{i \in U_X} \frac{V(r_i)}{l_{i,X}} N_0^i. \tag{5.5}$$

It follows that $(P_X)_{X \in \mathcal{B}}$ verifies (5.3), so it belongs to $\mathcal{R}_2$.

Next, consider a rate assignment $\mathbf{r} \in \mathbb{R}^{|U|}$ for which $\mathcal{F}_2 \neq \emptyset$. Let $\mathbf{P} \in \mathbb{R}^{|\mathcal{B}|}$ be a positive solution of (5.3). Define:

$$\tilde{P}_{iX} = \alpha V(r_i) P_X + V(r_i) \sum_{Y \in \mathcal{B} \setminus \{X\}} l_{i,X}^{Y} P_Y + V(r_i) l_{i,X}^{-1} N_0^i. \tag{5.6}$$

By simple substitution in (5.2) it can be shown that $\tilde{\mathbf{P}}$ is a solution of (5.2). ■

Lemma 5.2.1 implies that optimizing the rate assignment over $\mathcal{R}_1$ is equivalent to optimizing the rate assignment over $\mathcal{R}_2$, i.e., that $(P_{\mathcal{B}})$ given by

$$\max\{\sum_{i \in U} r_i : \mathbf{r} \in \mathcal{R}_2\}. \tag{$P_{\mathcal{B}}$}$$

is equivalent to (P).

The characterization of $\mathcal{R}_2$ is much more compact than the characterization of $\mathcal{R}_1$, as it uses the total powers per BTS instead of the powers assigned to all mobiles individually. The set of feasible rate assignments in $\mathcal{R}_2$ may be characterized via Perron-Frobenius theory. To this end, observe that system (5.3) can be rewritten as

$$\begin{cases} (I - T(\mathbf{r}))\mathbf{P} = \mathbf{c}(\mathbf{r}), \\ P_X^{max} - P_X \geq 0, \text{ for all } X \in \mathcal{B}, \end{cases}$$

with $T(\mathbf{r})$ the $N \times N$ matrix:

$$\begin{aligned} T(\mathbf{r})_{kk} &= \alpha \sum_{i \in U_{X_k}} V(r_i), \\ T(\mathbf{r})_{kj} &= \sum_{i \in U_{X_k}} V(r_i) l_{i,X_k}^{X_j}, k \neq j, k, j = 1, 2, \ldots, N, \end{aligned}$$

and

$$\mathbf{c}(\mathbf{r})_k = \sum_{i \in U_{X_k}} V(r_i) l_{i,X_k}^{-1} N_0^i.$$

Necessary and sufficient conditions for the existence of a positive power assignment satisfying (5.3) for a given rate assignment $\mathbf{r}$ can now be formulated in terms of the Perron-Frobenius eigenvalue $\lambda_{PF}(T(\mathbf{r}))$ of $T(\mathbf{r})$.

Lemma 5.2.2 *For a given rate assignment $\mathbf{r} \in \mathbb{R}^{|U|}$ with $r_{min} \leq r_i \leq R_{max}$, there exists a positive power assignment satisfying (5.3) if and only if $\lambda_{PF}(T(\mathbf{r})) < 1$ or, equivalently, if and only if all the principal minors of $I - T(\mathbf{r})$ are positive.*

Proof. Since $r_{min} > 0$, for a feasible rate allocation $\mathbf{r}$, $T(\mathbf{r})$ is irreducible, having only positive elements. From the theory of non-negative matrices [Sen73], it follows that for a given rate allocation $\mathbf{r} \in \mathbb{R}^{|U|}$, $(I - T(\mathbf{r}))\mathbf{P} = c(\mathbf{r})$ has a positive solution $\mathbf{P} \in \mathbb{R}^N$ for any vector $\mathbf{c}(\mathbf{r}) \in \mathbb{R}^N$, ($c(\mathbf{r}) \geq 0$ and $c(\mathbf{r}) \neq 0$) if and only if the Perron-Frobenius eigenvalue of the matrix $T(\mathbf{r})$ is strictly smaller than 1. This condition is equivalent to requiring all the principal minors of $I - T(\mathbf{r})$ to be positive [Sen73]. This implies that if $\mathcal{F}_2(\mathbf{r}) \neq \emptyset$, $\lambda_{PF}(T(\mathbf{r})) < 1$ and all the principal minors of $I - T(\mathbf{r})$ are positive. However, the opposite is not true, since

$\lambda_{PF}(T(\mathbf{r})) < 1$ only guarantees a positive solution of $(I - T(\mathbf{r}))\mathbf{P} = c(\mathbf{r})$, but not necessarily one that is bounded from above by the maximum transmission powers in the cells. ■

Remark 5.2.2 (Pole capacity) *A necessary condition for the existence of a feasible power assignment for a given rate assignment is that*

$$1 - \alpha \sum_{i \in U_X} V(r_i) > 0,$$

for all $X \in \mathcal{B}$.

For minimum rate assignment to all mobiles, this condition reads

$$|U_X| \leq W/(\alpha \epsilon_D r_{\min}) + 1,$$

where $|U_X|$ is the number of calls carried by BTS X. This determines the maximum number of calls that can be carried by BTS X (the pole capacity of BTS X). It is obvious that not all mobiles in a cell with

$$|U_X| \geq \frac{W}{\alpha \epsilon R_{max}} > 1,$$

can receive maximum rate.

5.3 Characterising optimal rates and powers

This section characterizes the optimal rate assignment for a given assignment of calls to the BTSs for the case of throughput optimization.

We first introduce some terminology for optimization problems (see [BNO03]). An inequality constraint that is satisfied with equality in a feasible point (solution) is said to be *active* in the feasible point. A feasible point is *regular* if the gradients of the active inequality constraints and the gradients of the equality constraints are linearly independent and *non regular* otherwise. For a power assignment $\mathbf{P} = (P_X)_{X \in \mathcal{B}}$, for two mobiles $i, j \in U_X$, we define the *received interference ordering* $\prec_{\mathbf{P}}$ as

$$i \prec_{\mathbf{P}} j \qquad \text{if} \sum_{Y \in \mathcal{B} \setminus X} l_{i,X}^Y P_Y + l_{i,X}^{-1} N_0^i < \sum_{Y \in \mathcal{B} \setminus X} l_{j,X}^Y P_Y + l_{j,X}^{-1} N_0^j. \tag{5.7}$$

Let $(\mathbf{P}^*, \mathbf{r}^*)$ be an optimal solution to $(P_{\mathcal{B}})$. As $(\mathbf{P}^*, \mathbf{r}^*)$ must satisfy (5.3), the rate assignment problem decomposes into rate assignment problems for the cells, i.e., for each cell $X \in \mathcal{B}$, $(r_i^*)_{i \in U_X}$ is the optimal solution of the following problem:

$$\max \sum_{i \in X} r_i$$
$$\text{such that} \quad \sum_{i \in U_X} V(r_i)(\alpha P_X^* + \sum_{Y \neq X} l_{i,X}^Y P_Y^* + l_{i,X}^{-1} N_0^i) = P_X^*,$$
$$r_{min} \le r_i \le R_{max}, i \in U_X.$$

Based on this decomposition, the next theorem states that if $(P_{\mathcal{B}})$ is feasible, in an optimal solution a call with lower received interference obtains a higher data rate. Furthermore, there is at most one mobile with intermediate rate, all other mobiles are served at rate r_{min} or R_{max}.

Theorem 5.3.1 *Let* $(\mathbf{P}^*, \mathbf{r}^*)$ *be an optimal solution to* $(P_{\mathcal{B}})$.
(a) For any two mobiles i *and* j *in the same cell, if* $i \prec_{\mathbf{P}^*} j$, *then* $r_i^* \ge r_j^*$.
(b) There exists an optimal power and rate assignment such that in each cell at most one mobile has intermediate rate r^*, $r_{min} < r^* < R_{max}$.

Proof. (a) For every $i \in U_X$, denote by

$$a_i = \sum_{Y \in \mathcal{B} \setminus \{X\}} l_{i,X}^Y P_Y^* + l_{i,X}^{-1} N_0^i.$$

Clearly, $i \prec_{\mathbf{P}^*} j$ is equivalent to $a_i < a_j$. Suppose that there exist two mobiles $i, j \in U_X$ such that $a_i < a_j$ and $r_i^* < r_j^*$. Since V is increasing,

$$V(r_i^*) < \frac{V(r_i^*)a_i + V(r_j^*)(a_j - a_i)}{a_j} < V(r_j^*).$$

Since V is continuous, there exists an α with $0 < \alpha < r_j^* - r_i^*$ such that

$$V(r_i^* + \alpha) = \frac{V(r_i^*)a_i + V(r_j^*)(a_j - a_i)}{a_j}.$$

Consider the rate assignment $\hat{\mathbf{r}}$ given by

$$\hat{r_k} = \begin{cases} r_k^*, \text{ for } k \in U_X \setminus \{i, j\} \\ r_j^*, \text{ for } k = i, \\ r_i^* + \alpha, \text{ for } k = j. \end{cases} \tag{5.8}$$

It is straightforward to check that the rate $(\hat{r}_i)_{i \in U_X}$ is feasible and since the utility function u is increasing, it has a higher total utility then $(r_i^*)_{i \in U_X}$. Hence, the optimality of $(r_i^*)_{i \in U_X}$ is contradicted. We conclude that $r_i^* \ge r_j^*$.

(b) Suppose that there exist two mobiles i and j with intermediate rate r_i^*, respectively r_j^* such that $r_{min} < r_j^* < R_{max}$ and $r_{min} < r_i^* < R_{max}$.

Suppose that $r_j^* < r_i^*$, and thus $a_i \leq a_j$.

Let

$$\alpha = \min\{R_{max} - r_i^*, r_j^* - r_{min}\}.$$

Assume that $\alpha = R_{max} - r_i^*$. Since V is concave,

$$V(R_{max}) - V(r_i^*) = V(r_i^* + \alpha) - V(r_i^*) < V(r_j^*) - V(r_j^* - \alpha),$$

and consequently

$$V(R_{max})a_i + V(r_j^* - \alpha)a_j < V(r_j^*)a_j + V(r_i^*)a_i < V(R_{max})a_i + V(r_j^*)a_j.$$

Since V is continuous and increasing, there exist a $0 < \beta < \alpha$ such that

$$V(R_{max})a_i + V(r_j^* - \beta)a_j = V(r_j)a_j + V(r_i)a_i.$$

One can easily verify that the rate assignment given by $\hat{\mathbf{r}}$ defined as

$$\hat{r_k} = \begin{cases} r_k^*, \text{ for } k \in U_X \setminus \{i,j\} \\ R_{max}, \text{ for } k = i, \\ r_j^* - \beta, \text{ for } k = j. \end{cases} \tag{5.9}$$

is feasible and it has a higher total rate than $\mathbf{r}^*$, since $\beta < \alpha$.

Similarly, if $\alpha = r_j^* - r_{min}$, one can show that there exists a β such that

$$r_j^* - r_{min} < \beta < R_{max} - r_i^*,$$

with the property that

$$\hat{r_k} = \begin{cases} r_k^*, \text{ for } k \in U_X \setminus \{i,j\} \\ r_i^* + \beta, \text{ for } k = i, \\ r_{min}, \text{ for } k = j. \end{cases} \tag{5.10}$$

is feasible and it has a higher total rate than than $\mathbf{r}^*$. Hence, we can conclude that there exists at most one mobile with intermediate rate in one cell. ■

Theorem 5.3.1 characterizes the rate assignment when the optimal powers are known. Below we will characterize the optimal joint rate and power assignment in a multi-cell environment with a maximal power per BTS. We first proceed with optimal solutions given by non regular points and then by regular points.

Theorem 5.3.2 *For each non regular point* $(\mathbf{P}, \mathbf{r})$ *at least one BTS* X *transmits at maximal power* $P_X^{\max}$.

Proof. Let $(\mathbf{P}, \mathbf{r})$ be a non regular point, feasible for $(P_{\mathcal{B}})$. Consider the matrix $M(\mathbf{P}, \mathbf{r})$ formed from the gradients of the equality constraints $(\nabla h_X(\mathbf{P}, \mathbf{r}))_{X \in \mathcal{B}}$ and by the gradients of the active inequality constraints. The matrix $M(\mathbf{P}, \mathbf{r})$ has thus the general form

$$M(\mathbf{P}, \mathbf{r}) = \begin{pmatrix} I - T(\mathbf{r}) & A(\mathbf{r}) \\ B(\mathbf{P}) & \mathbf{0} \\ \mathbf{0} & C(\mathbf{r}) \end{pmatrix},$$

where $T(\mathbf{r})$ was introduced in Section 3,

$$A(\mathbf{r}) = \begin{pmatrix} A_{X_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & A_{X_N} \end{pmatrix},$$

with $A_{X_i} = (-V'(r_i)(\alpha P_{X_i} + \sum_{Y \in \mathcal{B} \setminus \{X_i\}} l_{i,X_i}^Y P_Y + l_{i,X_1}^{-1} N_0^i))_{i \in U_{X_i}}$, $B(\mathbf{P})$ is the matrix obtained from the diagonal matrix $\overline{B}(\mathbf{P})$ with the diagonal

$$diag(\overline{B}(\mathbf{P})) = [I_{P_{X_k} = P_{X_k}^{max}}]_{\{X_k \in \mathcal{B}\}}$$

by deleting the zero rows. (Here $I_{\{a\}} = 1$ if a is true, and 0 otherwise). Furthermore, $C(\mathbf{r})$ is a matrix defined as:

$$C(\mathbf{r}) = \begin{pmatrix} C_1(\mathbf{r}) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & C_N(\mathbf{r}) \end{pmatrix},$$

with the sub matrices $C_k(\mathbf{r}) \in \mathbb{R}^{|\{i \in U_{X_k} : r_i = R_{max}\}|} \times \mathbb{R}^{|\{i \in U_{X_k}\}|}$ obtained from the diagonal square matrices with diagonal

$$diag(\overline{C_k}(\mathbf{r})) = [I_{\{r_i = R_{max}\}} - I_{\{r_i = r_{min}\}}]_{\{i \in U_X\}}$$

by deleting all rows for which the diagonal element equals zero.

Let K_r be the number of active inequality constraints of the form $r_{min} \leq r_i$ or $r_i \leq R_{max}$. Notice that since the principal minors of $I - T(\mathbf{r})$ are positive, $rank(I - T(r)) = N$.

Suppose that none of the base stations transmit at maximum power, in other words, all the constraints of the form $P_X \leq P_X^{max}$ are inactive. Then, matrix M has the following form:

$$M(\mathbf{P}, \mathbf{r}) = \begin{pmatrix} I - T(\mathbf{r}) & A(\mathbf{r}) \\ \mathbf{0} & C(\mathbf{r}) \end{pmatrix},$$

Since $rank(I - T(\mathbf{r})) = N$ and $C(\mathbf{r})$ has a single nonzero entry per row, $M(\mathbf{P}, \mathbf{r})$ has rank $K_r + N$, which implies that the rows of $M(\mathbf{P}, \mathbf{r})$ are linearly independent, a contradiction with the fact that $(\mathbf{P}, \mathbf{r})$ is non regular. We thus conclude that there must be at least one station transmitting at maximum power. ∎

Theorem 5.3.3 *Let $(\mathbf{P}^*, \mathbf{r}^*)$ be regular and a local optimum of problem $(P_{\mathcal{B}})$. Then there either exists a BTS that transmits with maximum power or $r_i^* = R_{max}$ for all mobiles $i \in U$.*

Proof. Consider the Lagrangian function corresponding to problem $P_{\mathcal{B}}$. Let $\lambda, \beta, \zeta \in \mathbb{R}^{|\mathcal{B}|}$ and $\mu,\ \nu \in \mathbb{R}^{|U|}$ be the Lagrangian multipliers corresponding to equations in (5.3), (5.4) and $r_{min} \leq R_{max}$. The Lagrangean corresponding to $P_{\mathcal{B}}$ is

$$\begin{aligned} L(\mathbf{P}, \mathbf{r}, \lambda, \beta, \zeta, \mu, \nu) \quad = \quad & \sum_{i \in U} r_i \\ & + \sum_{X \in \mathcal{B}} \lambda_X \left(\begin{array}{l} \left(1 - \alpha \sum_{i \in U_X} V(r_i)\right) P_X \\ - \sum_{Y \in \mathcal{B} \setminus \{X\}} \sum_{i \in U_X} V(r_i) l_{i,X}^Y P_Y \\ - \sum_{i \in U_X} V(r_i) l_{i,X}^{-1} N_0^i \end{array} \right) \\ & + \sum_{X \in \mathcal{B}} \beta_X (P_X^{max} - P_X) + \sum_{X \in \mathcal{B}} \zeta_X P_X \\ & + \sum_{i \in U} \mu_i (R_{\max} - r_i) + \sum_{i \in U} \nu_i (r_i - r_{\min}). \end{aligned}$$

The Karush-Kuhn-Tucker necessary conditions for a regular point $(\mathbf{P}^*, \mathbf{r}^*)$ that is an optimum of $(P_{\mathcal{B}})$ state that

there exists an unique multiplier vector $(\lambda^*, \beta^*, \zeta^*, \mu^*, \nu^*)$ such that:

(K1) $\bigtriangledown_{(\mathbf{P}^*, \mathbf{r}^*)} L(\mathbf{P}^*, \mathbf{r}^*, \lambda^*, \beta^*, \zeta^*, \mu^*, \nu^*) = 0$

(K2) $\beta^* \geq 0$, $\zeta^* \geq 0$, $\mu^* \geq 0$ and $\nu^* \geq 0$,

(K3) The Lagrangean multipliers corresponding to non active constraints are equal to 0.

Note that since the minimum rate can be ensured to all accepted mobiles, constraints (5.3) imply that $P_X^* > 0$ for each $X \in \mathcal{B}$. Thus, based on (K3), we conclude that $\zeta^* = \mathbf{0}$. Condition (K1) implies that $\frac{\partial L}{\partial P_X}(\mathbf{P}^*, \mathbf{r}^*, \lambda^*, \beta^*, \zeta^*, \mu^*, \nu^*) = 0$, and $\frac{\partial L}{\partial r_i}(\mathbf{P}^*, \mathbf{r}^*, \lambda^*, \beta^*, \zeta^*, \mu^*, \nu^*) = 0$. Combining with $\zeta^* = 0$ yields

$$\lambda^t(\mathbf{I} - T(\mathbf{r})) = \beta^*, . \tag{5.11}$$

Furthermore, for each mobile i, the following holds

$$1 + \nu_i^* - \mu_i^* - \lambda_X^* V'(r_i)(\alpha P_{X^*} + \sum_{Y \in \mathcal{B} \setminus \{X\}} l_{i,X}^Y P_Y^* + l_{i,X}^{-1} N_0^i) = 0. \tag{5.12}$$

Assume that none of the base stations transmits at maximum power and that there exists a mobile $i \in U$ such that $r_{min} \leq r_i < R_{max}$. From condition (K3) follows that $\beta^* = \mathbf{0}$. Since P^* is a positive solution of the system $(I - T(\mathbf{r}^*))P = c(r^*)$, by Lemma 5.2.2, $I - T(\mathbf{r}^*)$ is nonsingular. Hence, $\lambda^* = 0$.

Let i be the mobile with $r_{min} \leq r_i < R_{max}$. Clearly, $\mu_i^* = 0$. From (5.12) now follows that $\nu_i^* = -1$, which contradicts condition (K2).

We conclude that in a local optimal solution, either the rates of all mobiles are maximal, or the power in one of the cells is maximal. ■

Theorems 5.3.1 - 5.3.3 imply that in an optimal solution $(\mathbf{P}^*, \mathbf{r}^*)$ mobiles in a cell X are assigned R_{max}, r_{min}, or an intermediate rate r. This assignment is determined by the ordering $\prec_{\mathbf{P}^*}$, as summarized in the following corollary.

Corollary 5.3.4 *Let $(\mathbf{P}^*, \mathbf{r}^*)$ be an optimal solution of problem $(P_{\mathcal{B}})$. In each cell X there exists a mobile $i_k \in U_X$ such that*

$$\begin{cases} r_i = R_{max}, & \text{for all} \quad i \prec_{\mathbf{P}^*} i_k, \\ r_i = r_{min}, & \text{for all} \quad i \succ_{\mathbf{P}^*} i_k. \end{cases}$$

We have shown that in an optimal solution, there exists a BTS that transmits at maximal power. The following theorem refines this result.

Theorem 5.3.5 *Suppose that $N \geq 3$ and that there exists an optimal solution $(\mathbf{P}^*, \mathbf{r}^*)$ with $P_X^* = P_X^{max}$ for one BTS in $\mathcal{B}$. Then there either exists another BTS $Y \in \mathcal{B} \setminus \{X\}$ with $P_Y^* = P_Y^{max}$ or all mobiles assigned to BTSs in $\mathcal{B} \setminus \{X\}$ have maximal rate.*

Proof. Let $(\mathbf{P}^*, \mathbf{r}^*)$ be an optimal solution with $P_X^* = P_X^{max}$. Denote by $\tilde{\mathbf{P}} = (P_Y^*)_{Y \in \mathcal{B} \setminus \{X\}}$ and $\tilde{r} = (r_i)_{i \in \bigcup_{Y \in \mathcal{B} \setminus \{X\}} U_Y}$. Clearly, $(\tilde{\mathbf{P}}, \tilde{\mathbf{r}})$ is an optimal solution of the following mathematical program.

$$P_{\mathcal{B}\setminus\{\mathcal{X}\}}: \qquad \max \sum_{i\in U\setminus U_X} r_i$$

$$\left(1-\alpha\sum_{i\in U_{X'}} V(r_i)\right)P_{X'} - \sum_{Y\in\mathcal{B}\setminus\{X,X'\}}\sum_{i\in U_{X'}} V(r_i) l^{Y}_{i,X'} P_Y \tag{5.13}$$

$$= \sum_{i\in U_{X'}} V(r_i) l^{-1}_{i,X'}(N_0^i + \sum_{i\in U_X} \tilde{r}_i^* l^X_{i,X'} P_X^{max}),\ \text{for each } X' \in \mathcal{B}\setminus\mathcal{X}$$

$$P_{X'}^{max} - P_{X'} \geq 0, X' \in \mathcal{B}\setminus\{X\}$$

$$P_{X'} \geq 0, X' \in \mathcal{B}\setminus\{X\}$$

$$R_{max} - r_i \geq 0,\ \text{for } i \in U\setminus U_X,$$

$$r_i - r_{min} \geq 0,\ \text{for } i \in U\setminus U_X.$$

Note that $P_{\mathcal{B}\setminus\{\mathcal{X}\}}$ is a rate and power optimization problem for $N-1$ cells, with the noise for an arbitrary mobile $i \in U_{X'}$ defined by $N_0^i + \sum_{i\in U_X} r_i^* l^X_{i,X'} P_X^{max}$. Therefore, the results proven in Theorem 5.3.2 and Theorem 5.3.3 hold for $P_{\mathcal{B}\setminus\{\mathcal{X}\}}$. Hence, if $(\tilde{\mathbf{P}}, \tilde{\mathbf{r}})$ is a non regular point of $P_{\mathcal{B}\setminus\{X\}}$, then based on Theorem 5.3.2 we conclude that there must be a base station in $\mathcal{B}\setminus\{X\}$ transmitting at maximum power. On the other hand, if $(\tilde{\mathbf{P}}, \tilde{\mathbf{r}})$ is a regular point and an local optimum of $P_{\mathcal{B}\setminus\{X\}}$, Theorem 5.3.3 implies that either there exists a base station in $\mathcal{B}\setminus\{X\}$ transmitting at maximum power or all the mobiles corresponding to cells in $\mathcal{B}\setminus\{X\}$ have maximum rate. The statement of the theorem now follows. ■

Note that Theorems 5.3.2 and 5.3.3 imply that in an optimal solution, either all users receive R_{max}, or there is at least one BTS that transmits at maximal power. In the latter case, if $N > 3$, by following the steps in the proof of Theorem 5.3.5, one can reduce the problem to a rate and power assignment problem in $N-1$ cells. At its turn, this problem can be reduced to a problem in $N-2$,...3 cells. If $N = 3$, the problem reduces to a rate and power assignment problem for 2 cells. Based on Theorems 5.3.2 and 5.3.3, we obtain that either all users in the two cells get R_{max}, or one of the cells transmits at P^{max}. Note that in the last case, the rates in the other cell can take values $\{r_{min}, r, R_{max}\}$, with $r_{min} < r < R_{max}$. We have the following Corollary.

Corollary 5.3.6 *In a network with N cells, in the optimal power and rate assignment, only one of the following situations can occur:*

1. *all mobiles have R_{max};*
2. *in k cells all mobiles have R_{max}, the other $N-k$ BTSs transmit at maximal power;*

3. *$N-1$ BTSs transmit at maximal power.*

In the next sections we will give an exact algorithm for optimal power and rate assignment which is suitable for small networks and propose a heuristic for larger networks.

5.4 Exact algorithm for throughput maximization

This section provides an algorithm for optimal rate and power assignment to achieve maximal throughput. Although the algorithm is too slow for practical purposes, it is of interest as it provides an exact optimal rate and power assignment. This assignment will serve as benchmark for the heuristic power and rate assignment provided in Section 6.

The exact algorithm has four major steps based on Corollary 5.3.6. In Step 1 the algorithm assigns maximum rate to all mobiles. If this is infeasible, in Step 2 it assigns maximum rates to mobiles in $k < N-1$ cells and maximum power to the other cells. All the subsets of k elements of $\mathcal{B}$ will be checked. In Step 3 the algorithm analyzes the situation in which $N-1$ BTSs transmit at maximum power. In Step 4 the rate and power assignment with maximum throughput is chosen. The algorithm is summarized below.

Algorithm 1 The exact algorithm

Step 1: Assign maximum rate to all mobiles $i \in U$.
if There exists a feasible power allocation, **then**
  Return as optimal solution $r_i = R_{max}$ for all $i \in U$.
else
  Step 2: For all subsets $A = \{X_1, ..., X_k\} \subset \mathcal{B}$, with $k = 2, ..., N-1$ elements, assign R_{max} to the mobiles in A and maximum power to the mobiles in the cells in $\mathcal{B} \setminus A$. Find the rate and power assignment that gives maximum throughput.
  Step 3: For each cell $X \in \mathcal{B}$, assign maximum power to the cells in $\mathcal{B} \setminus X$. Calculate the power in X and the rate allocation that gives the maximum throughput.
  Step 4: Choose among the feasible rate and power allocations obtained in Step 2 and Step 3 the one that gives maximum tthroughput.
end if

Steps 1 and 4 are fairly obvious. Steps 2 and 3 require additional work. Next we will describe these steps in greater detail.

Step 2: maximum rates to mobiles in $k < N-1$ cells and maximum power to the other cells

Step 2 of the algorithm is based on two basic procedures. The first procedure calculates the powers in cells $A \subset \mathcal{B}$ when the powers in cells $\mathcal{B} \setminus A$ are known and the mobiles in the cells in A receive maximum rate. The second procedure subsequently calculates the optimal rates in cells in $\mathcal{B} \setminus A$. These procedures are described below.

Finding feasible powers when the rates are known.

Assume that all mobiles in the cells in A have maximal rate and the BTSs in $\mathcal{B} \setminus A$ transmit at maximal power. There exists a feasible power allocation for the cells in A if and only if the following system has a solution, recall (5.3):

$$\begin{cases} (1-\alpha \sum\limits_{i \in U_X} V(R_{max}))P_X - \sum_{Y \in A \setminus X} \sum\limits_{i \in U_X} V(R_{max}) l_{i,X}^Y P_Y \\ = \sum\limits_{i \in U_X} V(R_{max})(\sum_{Y \in \mathcal{B} \setminus A} l_{i,X}^Y P_Y^{max} + l_{i,X}^{-1} N_0^i), \text{ for each } X \in A, \\ 0 \leq P_X \leq P_X^{max}, \text{ for each } X \in A. \end{cases} \tag{5.14}$$

Following the arguments used to prove Lemma 5.2.2, it can be shown that feasibility of (5.14) implies unicity of its solution. Thus, we have determined the rate and power allocation for the cells and mobiles in A.

Finding the optimal rates when the powers are known.

When the powers in all cells are known, finding the optimal rate allocation for mobiles in $\mathcal{B} \setminus A$ reduces to solving $N-k$ rate allocation problems for which the power and the interference are known. Each such optimization problem corresponds to a cell $X \in \mathcal{B} \setminus A$ and has the following form:

$$\max\{\sum_{i \in X} r_i : \sum_{i \in U_X} V(r_i)(\alpha P_X + \sum_{Y \neq X} l_{i,X}^Y P_Y + l_{i,X}^{-1} N_0^i) = P_X, r_{min} \leq r_i \leq R_{max}, i \in U_X\}. \tag{$\mathcal{P}_X$}$$

By Theorem 4.2.5, in an optimal solution, if $i \preceq_{\mathbf{P}} j$, then $r_i \geq r_j$. Moreover, the rates belong to the set $\{r_{min}, r, R_{max}\}$, where $r_{min} < r < R_{max}$ and at most one mobile has intermediate rate. Hence, an optimal rate assignment in cell X can be obtained by ordering the mobiles in a cell according to $\preceq_{\mathbf{P}}$ and assigning R_{max} to mobiles as long as the power P_X is not exceeded, while all the mobiles have a rate greater than r_{min}. The intermediate rate is chosen such that the power P_X is reached. The exact algorithm is as follows.

Algorithm 2 Optimal rate allocation when powers are known

Step 1: Set $r_i = r_{min}$ for all $i \in U_X$.
Step 2: Order the mobiles in U_X such that $i_1 \preceq_{\mathbf{P}} ... \preceq_{\mathbf{P}} i_{|U_X|}$.
Step 3: Set in this order $r_i = R_{max}$ until for some index k setting $r_k = R_{max}$ yields $\sum_{i=1}^{|U_X|} V(r_i)(\alpha P_X + \sum_{Y \neq X} l_{i,X}^{Y} P_Y + l_{i,X}^{-1} N_0^i) > P_X$.
Step 4: Choose r_k such that $\sum_{i=1}^{|U_X|} V(r_i)(\alpha P_X + \sum_{Y \neq X} l_{i,X}^{Y} P_Y + l_{i,X}^{-1} N_0^i) = P_X$.

Step 3: $N-1$ BTSs transmit at maximum power

Let $X_2, ..., X_N$ be the cells in which the BTSs transmit at maximal power. In an optimal solution of the form $\mathbf{P} = (P_{X_1}, P_{X_2}^{max}, ..., P_{X_N}^{max})$, from Theorem 4.2.5, in each cell X_k there exists a mobile i_k such that $r_i = R_{max}$ if $i \prec_{\mathbf{P}} i_k$ and $r_i = r_{min}$ if $i \succ_{\mathbf{P}} i_k$. Moreover, there is at most one mobile with intermediary rate in each cell.

The definition of the order relation $\prec_{\mathbf{P}}$ implies that the ordering of the mobiles in cell X_1 is known (since the powers in cells $2, \ldots, N$ are maximal). For cells $X_2, \ldots, X_N$ this ordering depends on P_{X_1}. Fortunately, as implied by Theorem 5.4.1 below, there are at most $\sum_{k=1}^{N} |U_{X_k}|^2$ such orderings. In the sequel, let $\mathcal{PA}_{X_1}$ denote the set of power assignments in which all BTSs in $\mathcal{B} \setminus \{X_1\}$ transmit at maximum power.

Theorem 5.4.1 *Assume that in the optimal power and rate assignment $N-1$ BTSs transmit at maximum power. Let BTS X_1 be the BTS not transmitting at maximum power. There exists a partition of $[0, P_{X_1}^{max}]$ in intervals $L_1, ..., L_{K_{X_1}}$ such that for all power assignments $\mathbf{P} \in \mathcal{PA}_{X_1}$ with $P_{X_1} \in L_s$, for each pair of mobiles $i, j \in \mathcal{B} \setminus \{X_1\}$ in the same cell $i \succeq_{\mathbf{P}} j$ or $i \preceq_{\mathbf{P}} j$. Moreover, $K_{X_1} \leq \sum_{k=1}^{N} |U_{X_k}|^2$.*

Proof. Consider a power assignment $\mathbf{P} \in \mathcal{PA}_{X_1}$ and two mobiles i, j in cell $Y \in \mathcal{B} \setminus \{X_1\}$. Next we investigate for which values of P_{X_1}, $i \prec_{\mathbf{P}} j$.

Note that if $l_{i,Y}^{X_1} \neq l_{j,Y}^{X_1}$, then $i =_{\mathbf{P}} j$ for

$$P_{X_1} = \frac{l_{j,Y}^{-1} N_0^j - l_{i,Y}^{-1} N_0^i + \sum_{Z \in \mathcal{B} \setminus \{Y, X_1\}} (l_{j,Y}^{Z} - l_{i,Y}^{Z}) P_Z}{l_{i,Y}^{X_1} - l_{j,Y}^{X_1}}.$$

Define

$$b_{ij} = \frac{l_{j,Y}^{-1} N_0^j - l_{i,Y}^{-1} N_0^i + \sum_{Z \in \mathcal{B} \setminus \{Y, X_1\}} (l_{j,Y}^{Z} - l_{i,Y}^{Z}) P_Z}{l_{i,Y}^{X_1} - l_{j,Y}^{X_1}}.$$

By straightforward calculations based on the definition of $i \prec_{\mathbf{P}}$, we can verify that the following affirmations hold:

1. If $l_{i,Y}^{X_1} = l_{j,Y}^{X_1}$ then $i \preceq_{\mathbf{P}} j$ if $l_{j,Y}^{-1} N_0^j - l_{i,Y}^{-1} N_0^i + \sum_{Z \in \mathcal{B} \setminus \{Y, X_1\}} (l_{j,Y}^Z - l_{i,Y}^Z) P_Z \geq 0$ and $i \succ_{\mathbf{P}} j$ otherwise.
2. If $l_{i,Y}^{X_1} - l_{j,Y}^{X_1} > 0$ then $i \prec_{\mathbf{P}} j$ if $P_{X_1} < b_{ij}$ and $i \succ_{\mathbf{P}} j$ otherwise.
3. If $l_{i,Y}^{X_1} - l_{j,Y}^{X_1} < 0$ then $i \succ_{\mathbf{P}} j$ if $P_{X_1} < b_{ij}$ and $i \prec_{\mathbf{P}} j$ otherwise.
4. If $l_{i,Y}^{X_1} \neq l_{j,Y}^{X_1}$ then $i =_{\mathbf{P}} j$ if $P_{X_1} = b_{ij}$.

We can thus conclude that if $l_{i,Y}^{X_1} \neq l_{j,Y}^{X_1}$, $i \prec_{\mathbf{P}} j$ is influenced by the power in cell X_1 only through the sign of $P_{X_1} - b_{ij}$.

Consider further the sequence $(b_{ij})_{(i,j) \in \mathcal{B} \setminus \{X_1\}}$. Suppose that after sorting the sequence in increasing order, we obtain

$$b_{i_1 j_1} \leq b_{i_2 j_2} \leq ....b_{i_s j_s}.$$

Let $b_{i_m j_m}$ be the smallest positive term in the sequence and $b_{i_p j_p}$ be the largest term that that does not exceed $P_{X_1}^{max}$. The set of intervals

$$L_1 = [0, b_{i_m j_m}),$$
$$L_2 = [b_{i_m j_m}, b_{i_{m+1} j_{m+1}}), ..., L_{p-m+1} = [b_{i_{p-1} j_{p-1}}, b_{i_p j_p}), L_{p-m+2} = [b_{i_p j_p}, P_{X_1}^{max}]$$

form a partition of $[0, P_{X_1}^{max}]$.
Clearly, this partition has the desired properties. ∎

Theorem 5.4.1 implies that the problem of finding the optimal power assignment $\mathbf{P} \in \mathcal{PA}_{X_1}$ can be reduced to the problem of finding the power assignment $\mathbf{P} \in \mathcal{PA}_{X_1}$ with maximal throughput obtained when the power in cell X_1 is equal to the borders of the intervals $L_1, \ldots, L_{K_{X_1}}$, i.e., a finite set of powers. We will refer to the partition of $[0, P_{X_1}^{max}]$ that satisfies the conditions in Theorem 5.4.1 as the *partition associated* to cell X_1.

We may now describe Step 3 of the algorithm. Let $L = \{[0, b_{i_1 j_1}], ..., [b_{i_m j_m}, P_{X_1}^{max}]\}$ be the partition associated to X_1. From Theorem 5.4.1, for all $P_{X_1} \in L_s$ the ordering of the mobiles in cells $X_2, \ldots, X_N$ with respect to their received interference remains unchanged. Therefore, it suffices to consider the orderings given by $\mathbf{P} \in \mathcal{PA}_{X_1}$ with $P_{X_1} = b_{i_s j_s}$, $s = 1, ..., m$.

Suppose mobiles in each cell are ordered with respect to $\prec_{\mathbf{P}}$,
with $\mathbf{P} = (b_{i_s j_s}, P_{X_2}^{max}, ..., P_{X_N}^{max})$. Let i_k be the minimal index in cell k such that $r_{i_k} \neq R_{max}$. Note that for given vector $(i_1, ..., i_N)$, in each cell X_k the rates of all mobiles $j \neq i_k$ are either R_{max} or r_{min}. Hence, the goal is to find the vector $(i_1, ..., i_N)$ and the rates of the mobiles $i_1, ..., i_N$ such that the total throughput is

maximized. For a given i_1 with rate r the power assigned to cell X_1, denoted by $P_{X_1}(r)$, can be determined from (5.3) for cell X_1, and is given by

$$P_{X_1}(r) = \frac{\begin{array}{c}\sum_{j\in U_{X_1}\setminus\{i_1\}} V(r_j)\left(\sum_{k=2}^{N} l_{j,X_1}^{X_k} P_{X_k}^{max} + l_{j,X_1}^{-1} N_0^j\right)\\ +V(r)\left(\sum_{k=2}^{N} l_{i_1,X_1}^{X_k} P_{X_k}^{max} + l_{i_1,X_1}^{-1} N_0^{i_1}\right)\end{array}}{1-\alpha\left(\sum_{j\in U_{X_1}\setminus\{i_1\}} V(r_j) + V(r)\right)} \tag{5.15}$$

For a given vector $(i_1, .., i_N)$, all the rates in cell X_k are known except for the rate of mobile i_k. The rate of mobile i_k can be calculated as a function of the power in X_1 from the following equation (obtained from (5.3)):

$$V(\tilde{r}_k(r)) = \frac{P_{X_k}^{max} - \sum_{i\in X_k\setminus\{i_k\}} V(r_i)\left(\begin{array}{l}\alpha P_{X_k}^{max}\\ +\sum_{Y\in\mathcal{B}\setminus\{X_1,X_k\}} l_{i,X_k}^{Y} P_{Y}^{max}\\ +l_{i,X_k}^{Y} P_{X_1}(r)\end{array}\right)}{\alpha P_{X_k}^{max} + \sum_{Y\in\mathcal{B}\setminus\{X_1,X_k\}} l_{i_k,X_k}^{Y} P_{Y}^{max} + l_{i_k,X_k}^{Y} P_{X_1}(r)} \tag{5.16}$$

All the variables in $(P_{\mathcal{B}})$ are now either known or can be expressed as a function of the intermediary rate r of mobile i_1 in cell X_1, recall that $V(r) = \frac{\epsilon r}{W+\alpha\epsilon r}$. As a consequence, for a given ordering of the mobiles and a given vector $(i_1, ..., i_N)$, $(P_{\mathcal{B}})$ can be reduced to an optimization problem in $\mathbb{R}$.

Step 3 of the algorithm checks all vectors $(i_1, .., i_N)$. The number of vectors that will be checked by the algorithm can be considerably restricted as follows. Suppose the mobiles in each cell X_k are ordered according to $\prec_{\mathbf{P}}$, with $\mathbf{P} \in \mathcal{PA}_{X_1}$ and $P_{X_1} = b_{i_s j_s}$, $s \le m$. Consider the power assignment $\mathbf{P}' \in \mathcal{PA}_{X_1}$ with $P'_{X_1} = b_{i_{s+1} j_{s+1}}$. In each cell X_k do the following. Under both power assignments, assign R_{max} to mobiles in increasing order of their index in $\prec_{\mathbf{P}}$ until P_{X_k} would be exceeded. In order to maintain feasibility, assign an intermediary rate or r_{min} to the mobile i for which $r_i = R_{max}$ would render the solution infeasible. Let $i_k(\mathbf{P})$ and $i_k(\mathbf{P}')$ be the minimal indices of the mobiles for which $r \neq R_{max}$ under $\mathbf{P}$ and $\mathbf{P}'$, respectively. Since $P_{X_1} < P'_{X_1}$, it must be that $i_k(\mathbf{P}') > i_k(\mathbf{P})$. By the same reasoning, $i_k(\mathbf{P}') \ge i_k(\mathbf{P}(r)) \ge i_k(\mathbf{P})$, where $\mathbf{P}(r) \in \mathcal{PA}_{X_1}$ such that $P_{X_1} = P_{X_1}(r)$. As a consequence, it must be that $i_k \in [i_k(\mathbf{P}), i_k(\mathbf{P}')]$. A detailed description of Step 3 for the case $P_{X_1} = P_{X_1}^{max}$ for $i = 2, ..., N$ is as follows.

Algorithm 3 Step 3

Step 3.1: Order the mobiles in cell X_1 in increasing order of their pathloss
for $i \in X_1$ **do**
  Order the mobiles in cell X_1 in increasing order of their pl_{i1} values, where $pl_{i1} = \sum_{Y \in \mathcal{B} \setminus \{X_1\}} l^Y_{i,X_1} P^{max}_{X_k} + l^{-1}_{i,X_1} N^i_0$.
end for
Step 3.2: Find the partition associated to X_1
for all cells $k = 2, ..., N$ and all mobiles $i, j \in U_{X_k}$ with $l^{X_1}_{i,X_k} \neq l^{X_1}_{j,X_k}$ **do**
  $b_{ij} = \frac{l^{-1}_{j,X_k} N^j_0 - l^{-1}_{i,X_k} N^i_0 + \sum_{Y \in \mathcal{B} \setminus \{X_1, X_k\}} (l^Y_{j,X_k} - l^Y_{i,X_k}) P_Y}{l^{X_1}_{i,X_k} - l^{X_1}_{j,X_k}}$.
end for
Order the sequence (b_{ij}) in increasing order.
Consider the subsequence $0 \leq b_{i_1 j_1} \leq ... \leq b_{i_m j_m} < P^{max}_{X_1}$.
Set $b_{i_{m+1} j_{m+1}} = P^{max}_{X_1}$.
Step 3.3: Rate and power assignment for all the orderings determined by the partition associated to X_1
for $s = 1, ..., m+1$ **do**
  Set $P_{X_1} = b_{i_s j_s}$
  for $k = 2, ..., N$ and each mobile $i \in X_k$ **do**
    Calculate the pathloss $pl_{ik} = \sum_{Y \in \mathcal{B} \setminus \{X_k, X_1\}} l^Y_{i,X_k} P^{max}_{X_k} + l^{X_1}_{i,X_k} P_{X_1} + l^{-1}_{i,X_k} N^i_0$.
  end for
  Order the mobiles in cell X_k in increasing order of their pl_{ik} values.
  if $s \neq m+1$ **then**
    Find $i_k(\mathbf{P})$ and $i_k(\mathbf{P'})$,
    where $\mathbf{P} = (b_{i_s j_s}, P^{max}_{X_2}, ..., P^{max}_{X_N})$ and $\mathbf{P'} = (b_{i_{s+1} j_{s+1}}, P^{max}_{X_2}, ..., P^{max}_{X_N})$.
  end if
  for all $(i_1, ..., i_N) \in [i_1(\mathbf{P}), i_1(\mathbf{P'})] \times ... \times [i_N(\mathbf{P}), i_N(\mathbf{P'})]$ **do**
    for all cells $k = 1, ..., N$ **do**
      Set $r_i = R_{max}$ for all i with $pl_{ik} < pl_{i_k k}$
      Set $r_i = r_{min}$ for all i with $pl_{ik} > pl_{i_k k}$
    end for
    Solve the following optimization problem to find the intermediary rate in cell 1

$$\max\{r + \sum_{k=2}^{N} \tilde{r_k}(r) : P_{X_1}(r) \in [b_{i_s j_s}, b_{i_{s+1} j_{s+1}}], \tilde{r_k}(r) \in [r_{min}, R_{max}], r \in [r_{min}, R_{max}]\},$$

    with $P_{X_1}(r)$ defined by (5.15) and $\tilde{r_k}(r)$ chosen as to satisfy (5.16).
    for each mobile $u_k \in X_k$ **do**
      Set $r_k = \tilde{r}_k(r)$
    end for
  end for
end for

Remark 5.4.1 (Complexity) *The exact optimal power and rate assignment algorithm obtains the optimal solution, but is computationally intensive due to the following two reasons. First, Step 2 of the algorithm checks* $2^N - N - 1$ *partitions in cells that get* P^{max} *and cells that get* R_{max}*. Second, in Step 3, for each cell* X *which initially does not have maximum power,* $K_X \times |U_{X_1}| \times ... \times |U_{X_N}|$ *optimization problems may have to be solved, where* K_X *is the number of intervals in the partition associated to* $[0, P_X^{max}]$*. Note that the complexity of Step 3 is a consequence of the fact that we allow for an intermediary rate in the cell which does not have maximal power. In the next section we propose a heuristic which overcomes the computational shortcomings of Step 2 and Step 3 of the exact algorithm.*

5.5 Heuristic algorithm for the optimal rate and power assignment

The exact optimal power and rate assignment algorithm is computationally intensive and can therefore be used for small networks, only. This section proposes a heuristic power and rate assignment that also allows for optimization of large networks. The key idea of the heuristic is to replace Step 2 of the exact algorithm, where all partitions of $\mathcal{B}$ in 2 subsets are considered for assigning maximal power and maximal rate, by a procedure in which maximal power and maximal rate is assigned to cells/mobiles in a certain order. Based on the power assignment for the case when all mobiles have r_{min}, the heuristic first orders the cells in increasing order of the ratio between the interference they cause and the number of mobiles in the cell. Subsequently, the heuristic assigns P^{max} to k of the cells in this order, and R_{max} to the remaining $N - k$ cells ($k = 1, ..., N$). In this way, instead of checking $2^N - N - 1$ combinations for assigning P^{max} and R_{max}, only N combinations will be checked. We chose to order the cells with respect to the powers given by r_{min} since this rate assignment will give a set of feasible powers.

Step 3 will be fastened by not assigning an intermediary rate in the cell which does not have maximal power. If for example cell X is the cell in which initially the power is not maximal, the power in cell X can be easily determined if it is known which mobiles have maximal, respectively minimal rate. Subsequently, since all the powers are known, the problem can be reduced to solving N one cell rate assignment problems with known powers that can be solved with Algorithm 2. Note that due to the lack of the intermediate rate, in Step 3 there is no need to calculate the partitions (b_{ij}) associated to each cell and to solve the optimization problem associated to each partition. Finally, the assignment with maximum throughput among the assignments checked in Step 2 and Step 3 will be given as output. The heuristic is described in more detail below.

Step 1: Assign maximum rate to all mobiles $i \in U$.
if There exists a feasible power allocation, **then**
  Return as optimal solution $r_i = R_{max}$ for all $i \in U$.
else
  Step 2: Assign minimum rate to all mobiles and calculate the corresponding powers in each cell.
  For each cell X calculate the interference caused by BTS X to other cells relative to the number of mobiles in cell X: $\eta_X = \frac{\sum_{Y \in \mathcal{B} \setminus \{X\}} P_X \sum_{j \in U_Y} l_{j,Y}}{|U_X|}$.
  Order the BTSs $\mathcal{B}$ in increasing order of η_X.
  for $k = 1$ to N **do**
    Assign P^{max} to the cells $X_1, ..., X_k$.
    Assign R_{max} to all the mobiles in the cells $X_{k+1}, ..., X_N$
    Calculate the powers of the BTSs $X_{k+1}, ..., X_N$
    Calculate the rates in the cells $X_1, ..., X_k$
    Calculate the throughput obtained.
  end for
  Step 3: For each BTS X assign maximum power to all cells in $\mathcal{B} \setminus \{X\}$
  Order the mobiles in X in increasing order of $\sum_{Y \in \mathcal{B} \setminus \{X\}} l_{i,X}^Y P_Y^{max} + l_{i,X}^{-1} N_0^i$.
  for $i \in U_X$ (in the above order) **do**
    Set $r_j = R_{max}$ for $j \preceq_P i$ and $r_j = r_{min}$ for $j \succ_P i$.
    Find the power in cell X.
    Find the rates in cells $\mathcal{B} \setminus \{X\}$ by solving $N - 1$ one cell rate assignment problems with known powers.
    Calculate the throughput.
  end for
  Choose the feasible rate and power assignment which gives maximum throughput.
end if

In the next section we will evaluate the quality of the heuristic algorithm by comparison with the exact algorithm with respect to the total throughput and computation time.

5.6 Numerical results

This paper has provided an exact and a heuristic algorithm to optimize total downlink throughput in a W-CDMA system. It is shown that in each cell the exact solution allocates rates r_{min} and R_{max} to all mobiles except for a single mobile that receives rate r with $r_{min} < r < R_{max}$. A large share of the computation time of the exact algorithm is devoted to determining the mobile with intermediate rate and the value of this intermediate rate. The heuristic ignores this intermediate rate. This section investigates the accuracy and the speed-up of the heuristic in comparison with the exact solution. As the running time of the exact algorithm is substantial, we carry out the comparison for small networks, only.

The numerical examples use the parameters of W-CDMA as provided in [HT07]: the system chip rate W= 3.84 MHz, the orthogonality factor is 0.7, the path loss exponent is 4, the energy per bit to interference ratio threshold is 5dB, the thermal noise is -100 dBm/Hz, and the maximal transmit power for each BTS is $P^{max} =$ 20W. The downlink transmission rate is allowed to vary continuously between r_{min} = 32kbps and R_{max} = 384kbps, i.e., for feasibility all mobiles required the minimum rate of 32kbps. We consider 3 BTSs in a two dimensional flat area with the distance between BTSs equal to 1.5km. In all experiments, a pre-specified number of mobiles are generated in each cell and distributed according to a uniform distribution.

The exact and heuristic algorithms are coded in Matlab R2011a 64-bit for Mac, and ran on a MacBook Pro (Mac OS X Lion 10.7.1) with 2.4 GHz Intel Core 2 Duo processor and 4 GB 1067 MHz DDR3 memory.

5.6.1 Performance ratio and speed up factor

The performance ratio (ratio of throughput obtained by the exact algorithm and the heuristic algorithm) and the speed up factor (the ratio of the running time of the exact algorithm and the heuristic algorithm) characterize the performance of the heuristic. We first investigate the influence of the load for a network of 3 connected cells for 9 combinations of light, medium and high cell load to obtain insight in the load region where the heuristic performs good or bad. Then we consider the case of medium load and study the influence of the number of cells. Finally, we study the performance of the heuristic as function of the load in more detail.

For a three cell network, we consider 9 cases of load for the cells. Each cell may have low load (10 mobiles), medium load (30 mobiles), or high load (80 mobiles). For each of these cases, we generate the specified number of mobiles homogeneously distributed over the respective cells and run for each realization both the exact and the heuristic algorithm. Table I gives the average speed up factor and the average performance ratio with their standard deviations. From Table 5.1 we observe that the speed up factor of the heuristic is considerable, especially in the moderate load cases. For light load the speed up factor is low as most mobiles receive R_{max}, and similarly, for high load this factor is low as most mobiles receive r_{min}. The performance ratio of the heuristic is less than 7% indicating that only a few mobiles receive a different rate under the heuristic.

As a graphical illustration of the optimal rate profile and its heuristic approximation, Figures 5.1, 5.2 show the rates for all mobiles under the exact solution and

Table 5.1: *Evaluation heuristic for 9 load cases (L = 10, M = 30, H = 80 mobiles).*

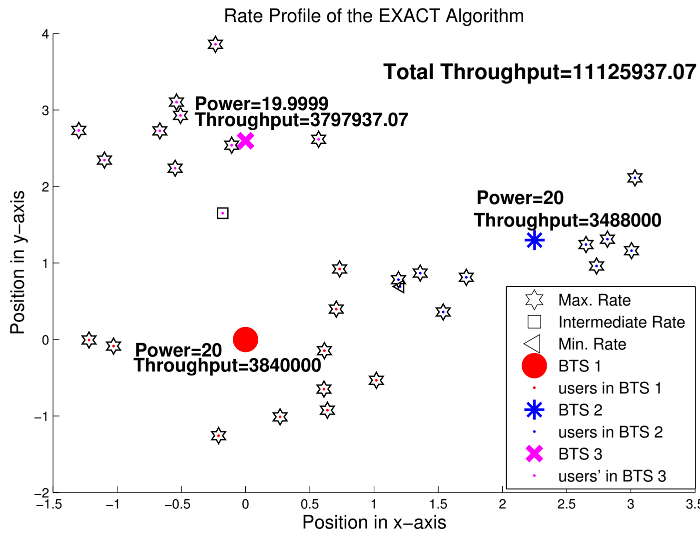
Case	speed up factor	performance ratio
(L,L,L)	36.89 (±1.82)	1.0548 (±0.0025)
(L,L,M)	108.24 (±3.69)	1.0549 (±0.0017)
(L,L,H)	617.97 (±16.21)	1.0593 (±0.0019)
(M,M,L)	141.95 (±3.66)	1.0559 (±0.0013)
(M,M,M)	155.91 (±3.74)	1.0543 (±0.0012)
(M,M,H)	387.21 (±14.32)	1.0589 (±0.0012)
(H,H,L)	390.67 (±15.59)	1.0624 (±0.0015)
(H,H,M)	296.83 (±10.55)	1.0665 (±0.0050)
(H,H,H)	104.42 (±8.82)	1.0795 (±0.0529)

our heuristic approximation for two instances. Indeed, almost all mobiles receive the same rate in both solutions. It is interesting to observe that the rate allocation is not monotone, that is, a mobile may receive rate R_{max} even though mobiles that are geographically closer to the BTS receive r_{min}, see BTS 3 in Figure 5.2. This is due to the ordering of mobiles according to $\prec_{\mathbf{P}}$ that takes into account the distance to the BTS serving the mobile *and* the other BTSs. As rates are assigned according to $\prec_{\mathbf{P}}$ to optimize throughput, this may result in a non geographically monotone rate allocation.

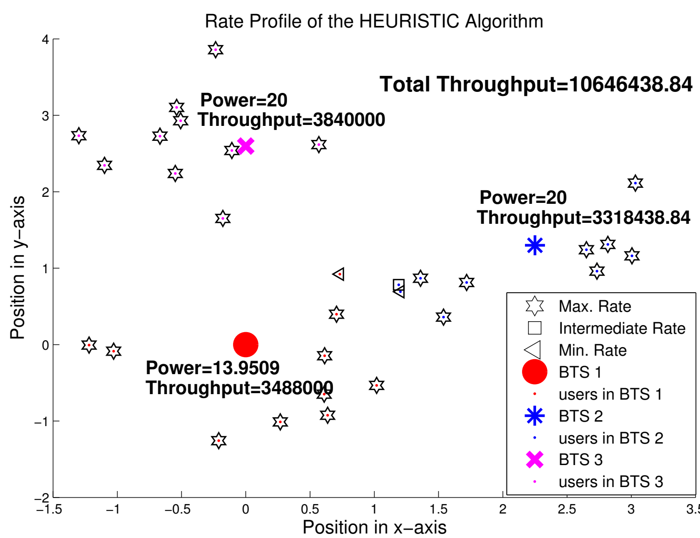
We now consider the impact of the number of cells on the performance of our heuristic. To this end, for moderate load (30 mobiles per cell) we consider a 7 cell circular network. Starting with the 3 cell network of a central cell and 2 adjacent cells in the circle around the central cell, we increase the network to the 7 cell network where all 6 cells in the circle are added. Table 5.2 gives the results. We observe a slight degradation of the performance of the heuristic.

Table 5.2: *Evaluation heuristic for multiple cells under medium load*

Case	speed up factor	performance ratio
3 cells	154.53 (±3.6)	1.05 (±0.012)
4 cells	135.10 (±2.6)	1.06 (±0.013)
5 cells	127.80 (±2.3)	1.06 (±0.014)
6 cells	124.50 (±1.9)	1.07 (±0.014)
7 cells	127.10 (±1.9)	1.07 (±0.014)

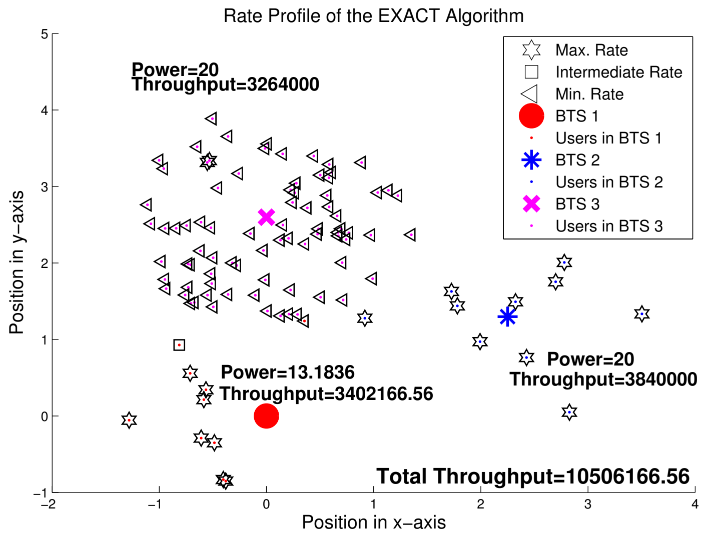


(a) Solved by the Exact Algorithm

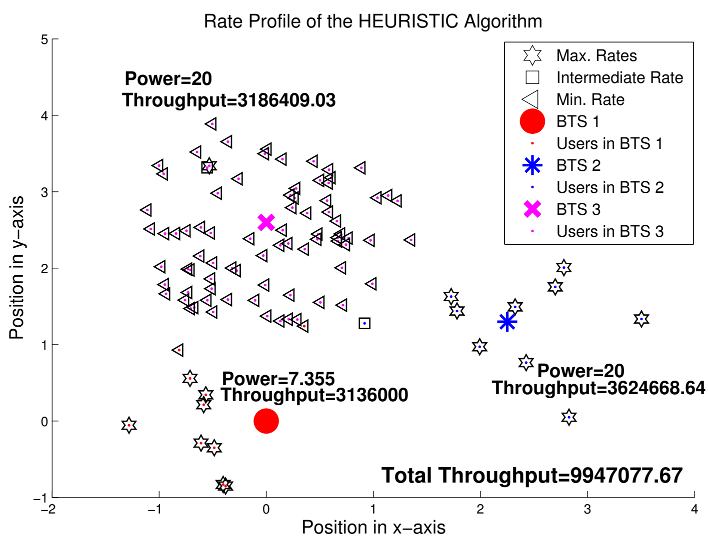


(b) Solved by Heuristic Algorithm

Figure 5.1: *Rate Profile for Case A with load [10,10,10]*



(a) Solved by the Exact Algorithm



(b) Solved by Heuristic Algorithm

Figure 5.2: *Rate Profile for Case C with load [10,10,80]*

We now consider the impact of increasing load. To this end, consider a 7 cell network with cell 1 the central cell, and cells $2-7$ in a ring around cell 1. We increase the load from light (10 mobiles) to medium (30 mobiles). Table III gives the results.

Table 5.3: *Evaluation heuristic for 7 cells under increasing load*

Case	speed up factor	performance ratio
(L,L,L,L,L,L,L)	27.6 ($\pm$0.7)	1.07 ($\pm$0.017)
(L,L,L,L,L,L,M)	47.6 ($\pm$0.9)	1.07 ($\pm$0.015)
(L,L,L,L,M,M,M)	71.4 ($\pm$1.3)	1.07 ($\pm$0.015)
(M,L,L,L,M,M,M)	85.2 ($\pm$1.5)	1.07 ($\pm$0.014)
(M,L,L,M,M,M,M)	110.5 ($\pm$1.9)	1.07 ($\pm$0.017)
(M,M,M,M,M,M,M)	129.1 ($\pm$2.0)	1.07 ($\pm$0.014)

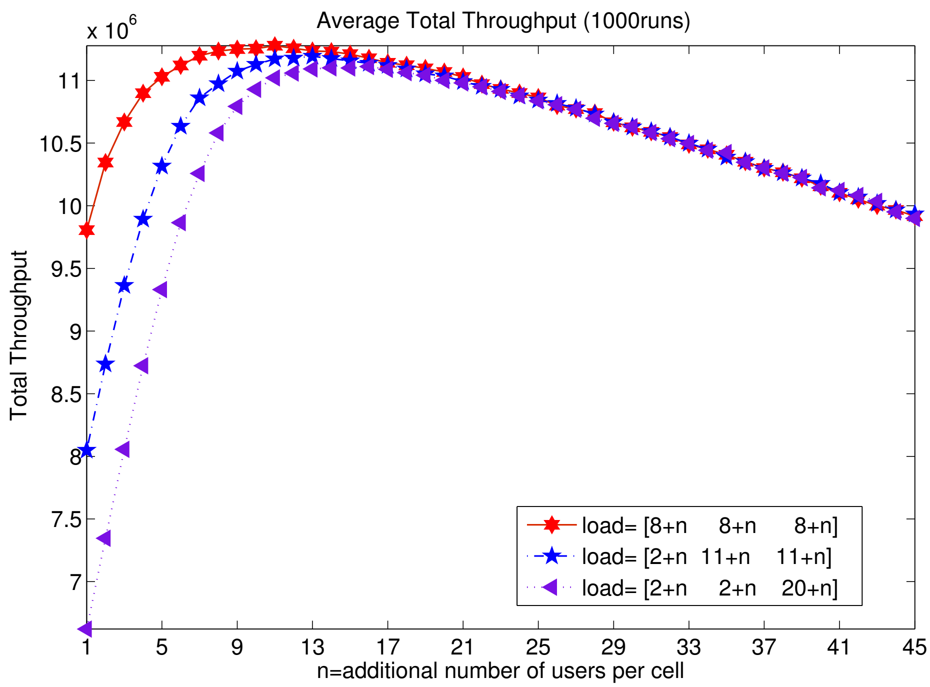
In conclusion, for medium load the heuristic performs good, and for light and high load the heuristic performs reasonably well. Note that the case of medium load is more relevant in practice as this is the range in which CDMA systems often operate.

5.6.2 The impact of cell load on the throughput

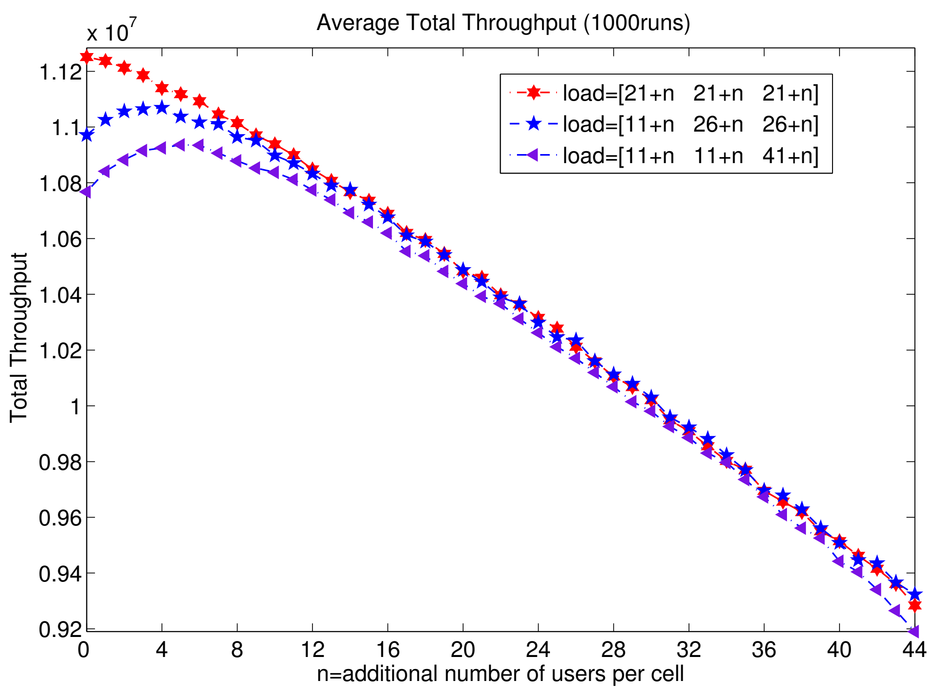
This section investigates the impact of the number of mobiles on the total throughput. We consider again a pre-specified number of mobiles per cell.

First, for three cases in which the total number of mobiles is the same, we consider the case where each cell contains $[8+n, 8+n, 8+n]$ mobiles, the case of two cells with high load, where the cells contain $[2+n, 11+n, 11+n]$ mobiles, and the case of a single cell with high load, where the cells contain $[2+n, 2+n, 20+n]$ mobiles, where n, $n = 0, 2, \ldots, 44$ is the additional number of mobiles in each cell. Figure 5.3(a) depicts the average total throughput as function of the number of additional mobiles, n.

For a different load combination, Figure 5.3(b) zooms in on the tail of the throughput curve for the cases with $[21+n, 21+n, 21+n]$ mobiles, $[11+n, 26+n, 26+n]$, and $[11+n, 11+n, 41+n]$ mobiles. We conclude, in agreement with intuition, that inhomogeneity in the load reduces the throughput. Furthermore, as the load increases the throughputs for different degrees of homogeneity tend to a common limiting curve when more and more mobiles are assigned r_{min}.



(a) Throughput versus load (light load)



(b) Throughput versus load (high load)

Figure 5.3: *Throughput versus load*

5.6.3 The impact of maximum rate on the throughput

The minimum and maximum rates clearly affect the total throughput. In the optimal solution (except for a single mobile in each cell) all mobiles are assigned either r_{min} or R_{max}. We now investigate the impact of R_{max} on the total throughput. Figure 5.4 considers three different values for R_{max} in a 3 cell network under homogeneous load with $8+n$ mobiles in each cell, $n = 1, \ldots, 62$, for R_{max} respectively 64kbps, 144 kbps and 384kbps. In agreement with intuition, for low load the impact of R_{max} on system throughput is high. For increasing load, however, this impact becomes smaller as more and more mobiles will be assigned r_{min}. Asymptotically, the impact of R_{max} becomes negligible. From the perspective of a network operator, this means that increasing R_{max} while maintaining r_{min} is optimal with respect to maximal total throughput.

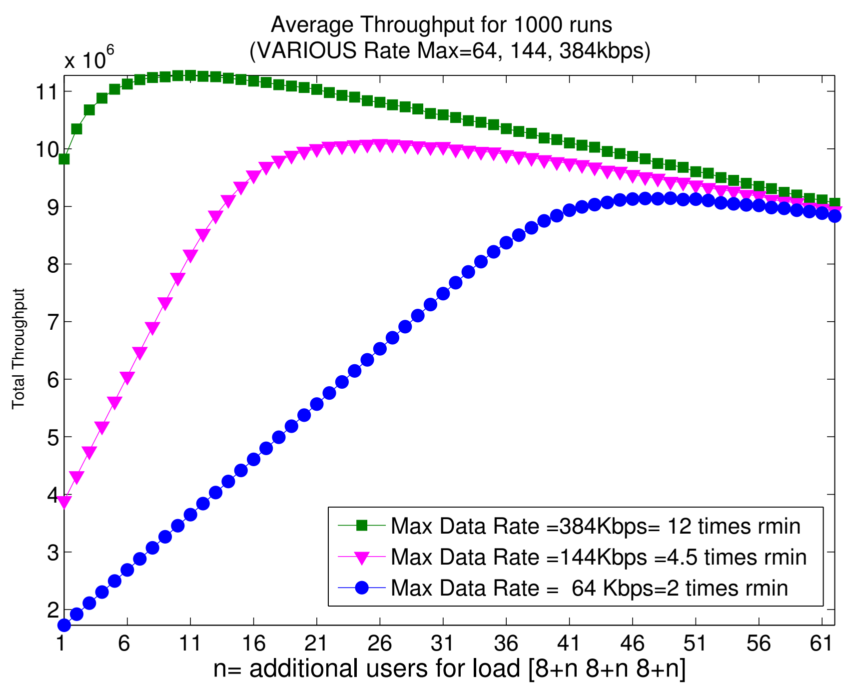


Figure 5.4: *Average Total Throughput of Various Load*

5.7 Conclusions

In this chapter, we have presented a joint downlink rate and power assignment for maximal total system throughput in a multi-cell CDMA network in an analytical setting. First, we derived an explicit and exact characterization of the structure of the optimal rate and power assignment: in a network with N base transmitter stations (BTSs) either all mobiles have maximum rate, or in k BTSs all mobiles have maximum rate and the other BTSs transmit at maximum power, or $N - 1$ BTSs transmit at maximum power. Second, we have given a characterization of the optimal rate assignment in each cell. Third, we have presented an exact algorithm for solving the rate and power assignment problem and a fast and accurate heuristic algorithm for power and rate assignment to achieve maximal downlink throughput in a multi-cell CDMA system. Via the numerical examples, we have shown that the heuristic algorithm is fast and close to exact. We have investigated the distribution of mobiles with high and low rate over the cells under various load scenarios and the dependence of throughput on cell load and the maximum rate parameter. Our results reveal that throughput maximization may be achieved by increasing R_{max}, but that at high load throughput reaches an asymptotic bound determined by the value of r_{min}.

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Summary

This thesis presents a full analytical characterization of the optimal joint downlink rate and power assignment for maximal total system throughput in a multi cell CDMA network.

In Chapter 2, we analyze the feasibility of downlink power assignment in a linear model of two CDMA cell, under the assumption that all downlink users in the system receive the same rate. We have obtained an explicit decomposition of system and user characteristics. Although the obtained relation is non-linear, it basically provides an effective interference characterisation of downlink feasibility for a fast evaluation of outage and blocking probabilities, and enable a quick evaluation of feasibility. We have numerically investigated blocking probabilities and have found for the downlink that it is best to allocate all calls to a single cell. Moreover, this chapter has also provided a model for determining an optimal cell border in CDMA networks. We have combined downlink and uplink feasibility model to determine cell borders for which the system throughput, expressed in terms of downlink rates, is maximized.

In Chapter 3, we have considered the two cell linear model where the coverage area was divided into small segments. Previously, we have assumed that all users in the cell are using the same rate, regardless their location. In this chapter, we have differentiated rate allocation based on their location. We have assumed that users in the same segment receive the same rate which is chosen from a discrete set. The goal is to assign rates to users in each segment, such that the utility of the system is maximized. In this chapter, we design an algorithm that is actually a fully polynomial time approximation scheme (FPTAS) for the rate optimization problem. The model in this chapter indicates that the optimal downlink rate allocation can be obtained in a distributed way: the allocation in each cell can be optimized independently, interference being incorporated in a single parameter t.

In Chapter 4, we have analyzed the two cell model under the assumption that the rates are continuous and may be chosen from a given interval. Moreover, we also taken into account the downlink limited transmit power. First, we developed a model for the joint rate and power allocation problem. Despite its non-convexity, the optimal solution in this chapter can be very well characterized. Second, we analyzed several properties of the optimal solutions. We have proved that the optimal rate allocations are monotonic as a function of the path loss. Based on this property, we have showed that in the optimal rate allocation, in each cell, only three rates are given to users. Finally, we have proposed a polynomial time algorithm in the number of users that solves optimally the joint rate and power allocation problem. The results can be extended to non-decreasing utility functions.

In Chapter 5, we have extended the model of the previous chapter to a multi-cell setting. We have presented a full analytical characterization of the optimal joint downlink rate and power assignment for maximal total system throughput in a multi cell CDMA network. Moreover, the cell model is a planar model. Chapter 5 has three main contributions. First, we provide an explicit and exact characterization of the structure of the optimal rate and power assignment. Second, we give a characterization of the optimal rate assignment in each cell. Third, based on these results, we give an exact algorithm for solving the rate and power assignment problem and a fast and accurate heuristic algorithm for power and rate assignment to achieve maximal downlink throughput in a multi cell CDMA system.

Irwan Endrayanto Aluicius

Curriculum Vitae

Irwan Endrayanto Aluicius was born in Klaten, Indonesia on October 28, 1972. He received his Bachelors degree in Mathematics at Gadjah Mada University, Indonesia in 1996. Afterwards, he became a staff member at the same department.

In 1998, via a bridging programe conducted by University of Twente in Institut Teknologi Bandung (ITB) Indonesia, he was awarded full scheme scholarship called TALIS (Talented Indonesian Students) from Netherlands Education Centre (NEC). From August 1998 until May 2000, he was a Master student at the Faculty of Applied Mathematics. Under supervision of Dr. Erik A. van Doorn (from Universiteit Twente) and Ir. Bart Sanders (from Libertel/Vodafone), he wrote a thesis with the title "Performance Modeling of CDMA Systems". He graduated in June 23, 2000 and obtained the title of Master of Science in Engineering Mathematics.

After returning home for one year, he went on September 2001 to became a PhD student (in Dutch: *Assistent in Opleiding* or AIO) at Stochastic Operation Research Group, Department of Applied Mathematics, University of Twente, under the supervision of prof. dr.Richard J Boucherie, prof. dr. J.L. van den Berg and Dr. Adriana F Gabor.

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Colophon

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